

Some relations between the largest eigenvalue and the frustration index of a signed graph

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Abstract

Let \dot{G} be a signed graph with n vertices and the frustration index ℓ . We prove the existence of k ($k \geq \ell$) edges e_1, e_2, \dots, e_k of \dot{G} such that

$$\lambda_1(\dot{G}) \leq \lambda_1(\dot{G} - e_1) \leq \lambda_1(\dot{G} - e_1 - e_2) \leq \dots \leq \lambda_1\left(\dot{G} - \sum_{i=1}^k e_i\right),$$

where λ_1 denotes the largest eigenvalue, and an inequality in above chain is strict unless the signed graph on the left hand side is disconnected with at least two components with the same largest eigenvalue. We also prove the existence of a connected balanced spanning subgraph \dot{H} such that

$$\ell \geq \frac{(\lambda_1(\dot{H}) - \lambda_1(\dot{G}))(\Delta + \lambda_1(\dot{H}))}{2\Delta} \sqrt{\frac{n}{n-1}},$$

where Δ is the maximum vertex degree in \dot{H} , along with the equality precisely in the trivial case when \dot{G} is balanced. We also derive some consequences of the previous results.

1 Introduction

We give a brief introduction; for more details, we refer the reader to Zaslavsky's survey [11]. A *signed graph* \dot{G} is a pair (G, σ) , where $G = (V, E)$ is an ordinary (unsigned) graph, called the *underlying graph*, and $\sigma: E \rightarrow \{-1, +1\}$ is the *signature*. In this context we do not make a distinction between the multiplicative groups $\{-1, +1\}$ and $\{-, +\}$. The edges mapped to 1 are *positive*, those mapped to -1 are *negative*, and together they comprise the

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edge set of \dot{G} ; we write this as $E^+(\dot{G}) \cup E^-(\dot{G}) = E(\dot{G})$. A graph G is interpreted as a signed graph with the all-positive signature; in our notation, it is recognized by the absence of a dot symbol. The number of vertices of \dot{G} is denoted by n . The $n \times n$ adjacency matrix $A_{\dot{G}}$ of \dot{G} is obtained from the standard adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. The eigenvalues, the spectrum and the eigenvectors of \dot{G} are the eigenvalues, the spectrum and the eigenvectors of $A_{\dot{G}}$, respectively. Since $A_{\dot{G}}$ is symmetric, the eigenvalues of \dot{G} are real. The largest eigenvalue is called the index and denoted by $\lambda_1(\dot{G})$.

A cycle in \dot{G} is *positive* if the product of its edge signs is 1; otherwise, it is *negative*. A signed graph is *balanced* if every cycle in it is positive; otherwise, it is *unbalanced*. Balance is a fundamental concept, and measuring how far a signed graph deviates from it is a useful information. An invariant that can be met in this context is the *frustration index* ℓ ; it is equal to the minimum number of edges whose deletion results in a balanced signed graph. Evidently, a largest balanced spanning subgraph of a signed graph is obtained by deleting ℓ edges. It is not necessarily unique. Computing the frustration index of a signed graph is a significant ingredient in solving problems in many fields including physics, chemistry, biology, social psychology and social networks. For more details, see [2] and references therein. For relations between the frustration index and some spectral invariants, we refer to [1, 3, 4, 7, 11].

Another fundamental concept is the switching equivalence. If U is a subset of the vertex set V of \dot{G} , then the signed graph \dot{G}^U obtained by reversing the sign of every edge with one end in U and the other in $V \setminus U$ is a *switching* of \dot{G} , and we say that \dot{G} and \dot{G}^U are *switching equivalent*. This concept is particularly important in the spectral theory since switching equivalent signed graphs share the same spectrum [11]. There is an approach in which the entire switching class is considered as a single signed graph [4]. For this paper it is important to know that for every eigenvalue of a signed graph, there exists a switching in which this eigenvalue is associated with a non-negative eigenvector [9].

In this study we consider the interplay between the index and the frustration index of a signed graph. The obtained relationships include the indices of certain spanning subgraphs and some structural invariants. Our results and the corresponding remarks are given in the next section. Their proofs are separated in Section 3.

2 Our contribution

If e is an edge of \dot{G} , then the spanning subgraph obtained by deleting e is denoted by $\dot{G} - e$. It is well-known (see [6, Proposition 1.3.10] or [8, Corollary 1.5]) that, for an unsigned graph G , $\lambda_1(G) > \lambda_1(G - e)$ holds for every $e \in E(G)$ whenever G is connected. Moreover, if G is disconnected, then there exists an edge for which the previous inequality holds, unless G has at least two components that share the same index. In the latter case we have $\lambda_1(G) = \lambda_1(G - e)$, for every $e \in E(G)$. In the context of signed graphs, deleting of an edge may result is a signed graph with a larger, or equal, or a smaller index. In short, anything is possible. In this paper we prove the following result.

Theorem 2.1. For a signed graph \dot{G} with the frustration index ℓ , there exist k ($k \geq \ell$) edges e_1, e_2, \dots, e_k such that

$$\lambda_1(\dot{G}) \leq \lambda_1(\dot{G} - e_1) \leq \lambda_1(\dot{G} - e_1 - e_2) \leq \dots \leq \lambda_1\left(\dot{G} - \sum_{i=1}^k e_i\right),$$

where an inequality in the above chain is strict unless the signed graph on the left hand side is disconnected with at least two components that share the same index.

In other words, we prove the existence of at least ℓ edges for which we have the inequalities that are opposite to those that occur in the context of graphs. Clearly, the interesting cases occur for $k \neq 0$. It is noteworthy to add that if \dot{G} is connected and balanced, then we have $k = 0$.

In the next result we estimate the index of a signed graph \dot{G} in terms of n, ℓ , and two invariants of a largest balanced spanning subgraph: its index and the maximum vertex degree. However, we formulate the result in the form of a lower bound for the frustration index, since computing the frustration index is NP-hard [10], while the quantities on the right hand side are computed in a polynomial time of n ; for the index, see [6].

Theorem 2.2. Let \dot{G} be a connected signed graph with n vertices and the frustration index ℓ , and \dot{H} a balanced subgraph obtained by deleting ℓ edges. Then \dot{H} is connected and

$$\ell \geq \frac{(\lambda_1(\dot{H}) - \lambda_1(\dot{G}))(\Delta + \lambda_1(\dot{H}))}{2\Delta} \sqrt{\frac{n}{n-1}}, \tag{2.1}$$

where Δ is the maximum vertex degree in \dot{H} . The equality holds if and only if \dot{G} is balanced.

For example, if we add a negative edge to the all-positive quadrangle, we obtain a signed graph with $\ell = 1$, while the right hand side of (2.1) is close to 0.49. Following the proof of the previous theorem (see the next section), one may obtain the inequality in which \dot{H} is replaced with G (the underlying graph of \dot{G}) and 2 in denominator is replaced with 4. The only difference in the proof occurs in (3.2) where both replacements are performed. Since $\lambda_1(\dot{H}) \geq \bar{d}$, where \bar{d} is the average vertex degree in \dot{H} (see [8, p. 28]), the inequality (2.1) remains valid if we write \bar{d} instead of the index of \dot{H} .

We also prove the following particular result. The *net degree* of a vertex in a signed graph is the difference between the number of positive and the number of negative edges incident with it. We say that a signed graph is *net-regular* if its net degree, considered as a function on the vertex set, is a constant.

Corollary 2.3. Let \dot{G} be a connected signed graph with n vertices and the frustration index ℓ . If a deletion of ℓ edges results in a balanced regular subgraph of vertex degree r , then

$$\ell \geq \frac{(r - \lambda_1(\dot{G}))n}{2}, \tag{2.2}$$

where the equality holds if and only if \dot{G} switches to a net-regular signed graph with net degree $\lambda_1(\dot{G}) = r - \frac{2\ell}{n}$.

This result is interesting since the equality may occur for $\ell > 0$. It also estimates the distance between the index of \dot{G} and the index of a largest balanced spanning subgraph in case when this subgraph is regular.

3 Proofs

If the vertices i and j are adjacent, then we write $i \sim j$. The existence of a positive (resp. negative) edge between these vertices is designated by $i \overset{+}{\sim} j$ ($i \overset{-}{\sim} j$). A *dominate component* of a signed graph \dot{G} is a component whose index coincides with the index of \dot{G} ; if \dot{G} is connected, then itself is the dominate component. Throughout the section we assume that an entry x_i of an eigenvector of \dot{G} corresponds to the vertex i . The all-positive unit eigenvector associated with the index of a connected graph is called the *principal eigenvector*. We use $\mathbf{0}$ and \mathbf{j} to denote the all-zero vector and the all-1 vector, respectively; the dimension will be clear from the context.

We proceed with the proof of our first result.

Proof of Theorem 2.1. Assume that $\ell \neq 0$, since otherwise there is nothing to prove. Let \dot{H} be a switching of \dot{G} in which the largest eigenvalue $\lambda_1(\dot{H})$ is associated with a non-negative unit eigenvector, say $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$; we have said in the introductory section that such a switching exists. If $e = ij$ is a negative edge of \dot{H} and \mathbf{y} is a unit eigenvector to $\lambda_1(\dot{H} - e)$, then we have

$$\begin{aligned} \lambda_1(\dot{H} - e) - \lambda_1(\dot{H}) &= \mathbf{y}^\top A_{\dot{H}-e} \mathbf{y} - \mathbf{x}^\top A_{\dot{H}} \mathbf{x} = \max_{\mathbf{z} \in \mathbb{R}^n, \|\mathbf{z}\|=1} \left(\mathbf{z}^\top A_{\dot{H}-e} \mathbf{z} \right) - \mathbf{x}^\top A_{\dot{H}} \mathbf{x} \\ &\geq \mathbf{x}^\top A_{\dot{H}-e} \mathbf{x} - \mathbf{x}^\top A_{\dot{H}} \mathbf{x} = \mathbf{x}^\top (A_{\dot{H}-e} - A_{\dot{H}}) \mathbf{x} = 2x_i x_j \geq 0. \end{aligned} \tag{3.1}$$

This proves that the desired inequality holds for $\lambda_1(\dot{G})$ and $\lambda_1(\dot{G} - e_1)$, where e_1 is any edge that becomes negative in the switching transformation of \dot{G} to \dot{H} ; since $\ell \neq 0$, such an edge must exist.

Clearly, this inequality reduces to equality if \dot{G} has at least two dominate components. Assume that there is exactly one dominate component. If \dot{H} is disconnected, then \mathbf{x} is zero on every vertex outside the dominate component.

We claim that \dot{H} has at least one negative edge for which at least one of the inequalities of (3.1) is strict. Assume for contradiction that this is not true. In this case, from (3.1) we deduce that $\lambda_1(\dot{H})$ appears in the spectrum of both \dot{H} and $\dot{H} - e$, and in both is afforded by \mathbf{x} .

If for some negative edge ij of \dot{H} we have $x_i = 0$ and $x_j \neq 0$, then by considering eigenvalue equations at the vertex i in both signed graphs we obtain

$$\lambda_1(\dot{H})x_i = \sum_{k \overset{+}{\sim} i} x_k - \sum_{k \overset{-}{\sim} i} x_k = \sum_{k \overset{+}{\sim} i} x_k - \sum_{\substack{k \overset{-}{\sim} i \\ k \neq j}} x_k - x_j = \lambda_1(\dot{H} - ij)x_i - x_j = \lambda_1(\dot{H})x_i - x_j,$$

which is impossible for $x_j \neq 0$.

Suppose now that for every negative edge ij we have $x_i = 0, x_j = 0$. First, the dominate component of \dot{H} has at least one positive edge, as otherwise we would have $\mathbf{x} = \mathbf{0}$. Consequently, this component contains a vertex, say s , that is incident with positive and negative edges. Again, from the eigenvalue equation and the fact that \mathbf{x} has no negative entries, we obtain $x_k = 0$ for every $k \sim s$. Proceeding in this way, we arrive at $x_k = 0$, for $1 \leq k \leq n$, which is impossible.

Therefore, there exists at least one negative edge satisfying at least one strict inequality. Denoting such an edge by e_1 , we obtain $\lambda_1(\dot{G}) = \lambda_1(\dot{H}) < \lambda_1(\dot{H} - e_1) = \lambda_1(\dot{G} - e_1)$, as desired.

To conclude the proof, we replace \dot{G} with $\dot{G} - e_1$ in the beginning and repeat the previous part to obtain the second inequality in the statement formulation. This procedure is repeated as long as \dot{H} has at least one negative edge. Since the frustration number of \dot{G} is ℓ , it will be performed at least ℓ times. To see this, it is sufficient to observe that a deletion of less than ℓ edges of \dot{G} results in an unbalanced signed graph, and so every its switching has at least one negative edge. The proof is completed. \square

To prove Theorem 2.2, we need the following lemmas. The first one is a known result.

Lemma 3.1 (Cioabă and Gregory [5, Theorem 3.2 and Lemma 3.3]). *Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ be the principal eigenvector of a connected graph G with n ($n \geq 2$) vertices, the maximum vertex degree Δ and the minimum vertex degree δ . For $x_p = \max_{1 \leq i \leq n} x_i$, we have*

$$\frac{\Delta}{(n-1)\Delta + \delta} \leq x_p^2 \leq \frac{\Delta}{\Delta + \lambda_1(G)},$$

with equality in the second place if and only if p is adjacent to all vertices of a regular graph on $n - 1$ vertices.

We continue with a particular case.

Lemma 3.2. *Let $\mathbf{x} = (x_1, x_2, \dots, x_n)^\top$ be the principal eigenvector of a connected graph G with n ($n \geq 2$) vertices and the maximum vertex degree Δ . For $x_q = \max_{i \neq p, i \sim p} x_i$, we have*

$$x_q^2 \leq \frac{\Delta(n-1)}{(\Delta + \lambda_1(G))n}.$$

Proof. We set $\lambda_1 = \lambda_1(G)$ and use d_i to denote the degree of a vertex i . We have the following chain of equalities and inequalities (for a brief explanation, see the subsequent text):

$$\begin{aligned} 1 &= \sum_{i=1}^n x_i^2 = x_p^2 + x_q^2 + \sum_{i \notin \{p,q\}} x_i^2 \geq x_p^2 + x_q^2 + \sum_{i \sim q} x_i^2 \geq x_p^2 + x_q^2 + \frac{1}{d_q} \left(\sum_{i \sim q} x_i \right)^2 \\ &= x_p^2 + x_q^2 + \frac{x_q^2 \lambda_1^2}{d_q} \geq \frac{\Delta}{(n-1)\Delta + \delta} + \left(1 + \frac{\lambda_1^2}{\Delta} \right) x_q^2. \end{aligned}$$

Precisely, the second inequality in the previous chain follows from the Cauchy-Schwarz inequality, as

$$d_q \sum_{i \sim q} x_i^2 = \left(\sum_{i \sim q} 1^2 \right) \left(\sum_{i \sim q} x_i^2 \right) \geq \left(\sum_{i \sim q} x_i \right)^2,$$

while the third inequality follows from Lemma 3.1 and $d_q \leq \Delta$. Now, we have

$$x_q^2 \leq \frac{\Delta}{\Delta + \lambda_1^2} \left(1 - \frac{\Delta}{(n-1)\Delta + \delta} \right),$$

which, after replacing δ with Δ , leads to the desired result. \square

Now, the announced proof.

Proof of Theorem 2.2. First, we prove that \dot{H} is connected. Although we believe that this is known, for the sake of completeness we provide a short argumentation. We delete the ℓ edges in question one-by-one, and in every step observe that the edge that should be deleted belongs to at least one cycle (even more, at least one negative cycle) of the obtained subgraph. Since this edge is in a cycle, its deletion does not result in a disconnected subgraph.

Since \dot{H} is balanced, without loss of generality we may assume that it has no negative edges (i.e., it is isomorphic to its underlying graph H). In this case, an edge of \dot{G} is positive if and only if it belongs to H . To see this it is sufficient to observe that the existence of a negative edge outside H implies the existence of a balanced spanning subgraph obtained by deletion of less than ℓ edges of \dot{G} , which is impossible.

Let \mathbf{y} be the principal eigenvector of H . As in the proof of Theorem 2.1, we compute

$$\lambda_1(H) - \lambda_1(\dot{G}) \leq \mathbf{y}^\top (A_H - A_{\dot{G}}) \mathbf{y} = 2 \sum_{ij \in E^-(\dot{G})} y_i y_j. \quad (3.2)$$

Since i and j are non-adjacent in H , from Lemmas 3.1 and 3.2, we obtain

$$y_i y_j \leq \sqrt{\frac{\Delta}{\Delta + \lambda_1(H)}} \sqrt{\frac{\Delta(n-1)}{(\Delta + \lambda_1(H))n}} = \frac{\Delta}{\Delta + \lambda_1(H)} \sqrt{\frac{n-1}{n}},$$

for all $ij \in E^-(\dot{G})$. This leads to

$$\lambda_1(H) - \lambda_1(\dot{G}) \leq 2\ell \frac{\Delta}{\Delta + \lambda_1(H)} \sqrt{\frac{n-1}{n}},$$

and the desired inequality follows.

If \dot{G} is balanced, then we have $\ell = 0$ and $\lambda_1(\dot{H}) = \lambda_1(\dot{G})$, and so the equality in (2.1) holds. Conversely, if \dot{G} is unbalanced and we have equality in (2.1), then we have equality in many places, but in particular there is the equality in the upper bound of Lemma 3.1. If this equality holds, then the corresponding vertex (denoted by p in the lemma) must have degree $n - 1$ in H , but it has not since it is incident with at least one negative edge in \dot{G} . This contradiction concludes the proof. \square

Finally, we prove the corollary.

Proof of Corollary 2.3. Suppose a deletion of ℓ edges of \dot{G} results in a balanced regular subgraph \dot{H} of vertex degree r . In this case, r is the index of \dot{H} , see [8, p. 11]. As in the proof of Theorem 2.2, we may apply a switching that transforms \dot{H} into H , and simultaneously \dot{G} into a switching equivalent signed graph, say \dot{G}' . The principal eigenvector of H is $\frac{1}{\sqrt{n}} \mathbf{j}$ and, in this case, the inequality (3.2) becomes $r - \lambda_1(\dot{G}') \leq \frac{2\ell}{n}$, which leads to (2.2) as $\lambda_1(\dot{G}) = \lambda_1(\dot{G}')$.

It remains to consider the equality in (2.2). Assume first that this equality occurs. Then, as in the previous proofs, we have the equality in (3.2), which means that $\frac{1}{\sqrt{n}} \mathbf{j}$ is an eigenvector to $\lambda_1(\dot{G}')$. We know from [7] that a signed graph has a constant eigenvector if and only if

it is net-regular (and then this eigenvector is associated with the net degree). Since H is regular, and \dot{G}' is net-regular, we deduce that negative edges of \dot{G}' induce a regular signed graph of degree $\frac{2\ell}{n}$, which gives the net degree $r - \frac{2\ell}{n}$. Since $\lambda_1(\dot{G}')$ and the net degree share the same eigenvector, they must coincide. Summa summarum, \dot{G} switches to a net-regular signed graph with net degree $\lambda_1(\dot{G}) = r - \frac{2\ell}{n}$, as desired.

Conversely, if \dot{G} is as in the statement formulation, then the equality in (2.2) follows by a direct computation. \square

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