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New formulations of the union-closed sets conjecture

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Abstract

The union-closed sets conjecture states that if a finite set $\mathcal{A}$ of finite sets is union-closed and $\mathcal{A} \neq \{\varnothing\}$, then there exists an element in $\cup_{A\in\mathcal{A}} A$ that belongs to at least half of the sets in $\mathcal{A}$. We present three new formulations of the union-closed conjecture in terms of matrices, graphs, and hypergraphs.

1 Introduction

The union-closed sets conjecture is a famous open problem in set theory. It is also known as the Frankl's conjecture because Peter Frankl formulated the conjecture in 1979 [2]. A collection $\mathcal{A}$ of sets is *union closed* if the union of any two sets in $\mathcal{A}$ is also a set in $\mathcal{A}$.

Conjecture 1.1 (The union-closed sets conjecture)**.** Let $\mathcal{A}$ be a finite set of finite sets. If $\mathcal{A} \neq \{\varnothing\}$ and $\mathcal{A}$ is union-closed, then there exists an element in $\cup_{A\in\mathcal{A}} A$ that belongs to at least half of the sets in $\mathcal{A}$.

In this article we will sometimes abbreviate the union-closed sets conjecture by "the UCSC". Note that union-closure of sets is a sufficient condition, not a necessary one, for an element in the union of the sets to belong to at least half of the sets. The following is another sufficient condition:

Observation 1.2. Let $\mathcal{A} = \{S_1, S_2, \ldots, S_n\} \neq \{\varnothing\}$ be a finite set of finite sets (not necessarily union-closed). If

$$\sum_{i=1}^{n} |S_i| \geq \frac{n}{2} \left| \bigcup_{i=1}^{n} S_i \right|,$$

then there exists $x \in \cup_{i=1}^{n} S_i$ that belongs to at least half of the sets in $\mathcal{A}$.

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Proof. Suppose $\cup_{i=1}^{n} S_i = \{x_1, x_2, \ldots, x_m\}$ and $\mathrm{d(x_i)}$ denotes the number of sets in $\mathcal{A}$ containing x_i. Then $\sum_{i=1}^{n} |S_i| = \sum_{j=1}^{m} \mathrm{d(x_j)}$. Suppose

$$\sum_{i=1}^{n} |S_i| \geq \frac{n}{2} \left| \bigcup_{i=1}^{n} S_i \right| = \frac{nm}{2}.$$

Then $\sum_{j=1}^{m} \mathrm{d(x_j)} \geq \frac{\mathrm{nm}}{2}$ which implies $\mathrm{d(x_j)} \geq \frac{\mathrm{n}}{2}$ for some $j = 1, 2, \ldots, m$. Thus x_j belongs to at least half of the sets in $\mathcal{A}$. □

The UCSC is true for the following cases:

1. $|\mathcal{A}| \leq 46$ (see [4], [6]).
2. $| \cup_{A \in \mathcal{A}} A| \leq 11$ (see [1])
3. $\mathcal{A}$ has a set of size 1 or 2 (see [7]).

For more detailed literature on the USCS, we refer [5, 3]. Despite the simple statement of the UCSC, it has been open for more than four decades. To prove the conjecture, we may need some tools from other fields such as graph theory and linear algebra. Therefore we present three new equivalent formulations of the UCSC in terms of matrices, graphs, and hypergraphs. Although these new formulations may not change the complexity of a possible proof of the conjecture, they will hopefully bring some insights to it.

2 Matrix formulation

In this section we present a formulation of the UCSC in terms of a certain property of columns of a binary matrix. We denote the binary field by $\mathbb{F}_2$. The Hadamard (or entrywise) product of two vectors $C_i, C_j \in \mathbb{F}_2^m$ is denoted by $C_i \circ C_j$.

Definition 2.1. A set $S = \{C_1,\ C_2, \ldots, C_n\}$ of n vectors in $\mathbb{F}_2^m$ is *addition-closed* if $(C_i + C_j + C_i \circ C_j)_2$ is in S for all $i, j = 1, 2, \ldots, n$.

We denote $(C_i + C_j + C_i \circ C_j)_2$ by $C_i \oplus C_j$ for simplicity.

Example 2.2. The columns of the following matrix M are addition-closed.

$$M = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Observation 2.3. An addition-closed set S of n vectors in $\mathbb{F}_2^m$ including the zero vector forms a monoid under $\oplus$ where the zero vector is the identity.

Conjecture 2.4 (The addition-closed columns conjecture)**.** Suppose M is an $m \times n$ binary matrix with nonzero rows and with distinct columns possibly including a zero column. If the columns of M are addition-closed, then M has a row with at least $\frac{n}{2}$ 1s.

Theorem 2.5. *The addition-closed columns conjecture (Conjecture 2.4) is equivalent to the union-closed sets conjecture (Conjecture 1.1).*

Proof. Suppose the union-closed sets conjecture (Conjecture 1.1) is true. Let

$$M = [C_1\; C_2 \cdots C_n]$$

be an $m \times n$ binary matrix with nonzero rows, distinct columns, and at most one zero column. Suppose the columns of M are addition closed. Construct n sets $S_1, S_2, \ldots, S_n$ that are subsets of $\{1, 2, \ldots, m\}$ such that $i \in S_j$ if and only if (i, j)-entry of M is 1. Let $\mathcal{A} = \{S_1, S_2, \ldots, S_n\}$. Since nonzero matrix M has distinct columns and at most one zero column, $\mathcal{A}$ is a set and $\mathcal{A} \neq \{\varnothing\}$. Since M has no zero rows, $\cup_{i=1}^{n} S_i = \{1, 2, \ldots, m\}$. Let $S_i, S_j \in \mathcal{A}$. Note that nonzero entries of

$$C_i \oplus C_j = (C_i + C_j + C_i \circ C_j)_2$$

are precisely indexed by $S_i \cup S_j$. Since the columns of M are addition closed, $C_i \oplus C_j$ is a column of M and consequently $S_i \cup S_j \in \mathcal{A}$. Thus $\mathcal{A}$ is union-closed. By the union-closed sets conjecture (Conjecture 1.1), there exists $k \in \cup_{i=1}^{n} S_i = \{1, 2, \ldots, m\}$ that belongs to at least half of the sets in $\mathcal{A}$ and consequently row k of M has at least $\frac{n}{2}$ 1s. Thus the addition-closed columns conjecture is true.

Conversely suppose the addition-closed columns conjecture (Conjecture 2.4) is true. Let $\mathcal{A} = \{S_1, S_2, \ldots, S_n\} \neq \{\varnothing\}$ be a finite set of finite sets that is union-closed. Suppose $\cup_{i=1}^{n} S_i = \{1, 2, \ldots, m\}$. Construct an $m \times n$ binary matrix $M = [m_{ij}]$ such that $m_{ij} = 1$ if and only if $i \in S_j$. Note that M has nonzero rows, distinct columns, and at most one zero column. Let

$$M = [C_1\; C_2 \cdots C_n].$$

Consider columns C_i and C_j of M. Note that nonzero entries of

$$C_i \oplus C_j = (C_i + C_j + C_i \circ C_j)_2$$

are precisely indexed by $S_i \cup S_j$. Since $\mathcal{A} = \{S_1, S_2, \ldots, S_n\}$ is union-closed, $S_i \cup S_j = S_k \in \mathcal{A}$ for some $k = 1, 2, \ldots, n$ and consequently $C_k = C_i \oplus C_j$. Thus columns of M are addition-closed. By the addition-closed columns conjecture (Conjecture 2.4), M has a row, say row t, with at least $\frac{n}{2}$ 1s and consequently $t \in \{1, 2, \ldots, m\} = \cup_{i=1}^{n} S_i$ belongs to at least half of the sets in $\mathcal{A} = \{S_1, S_2, \ldots, S_n\}$. Thus the union-closed sets conjecture is true. □

3 Graph and hypergraph formulations

One graph formulation of the UCSC was given in terms of maximal stable sets in [2]. Here we present a different one by introducing a special kind of bipartite graphs.

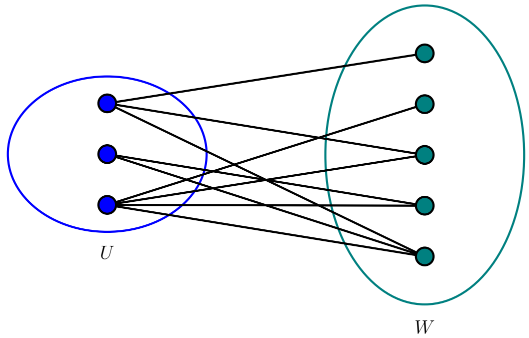

Figure 1: A union-closed bipartite graph with the vertex set $U\dot{\cup}W$

Definition 3.1. A bipartite graph G with the vertex set $U\dot{\cup}W$ is *union-closed* if for any vertices $w_i, w_j \in W$, there exists a vertex $w_k \in W$ such that

$$\mathrm{N(w_i)} \cup \mathrm{N(w_j)} = \mathrm{N(w_k)},$$

where $\mathrm{N(w)}$ denotes the neighborhood of a vertex w.

Example 3.2. A union-closed bipartite graph with the vertex set $U\dot{\cup}W$ is given in Figure 1.

Conjecture 3.3 (The union-closed graphs conjecture)**.** Suppose G is a finite bipartite graph with at least one edge and with the vertex set $U\dot{\cup}W$ where $U = \sum_{w\in W} \mathrm{N(w)}$ and W has no duplicate vertices. If G is union-closed, then there exists a vertex $u \in U$ of degree at least $\frac{1}{2}|W|$.

Theorem 3.4. *The union-closed graphs conjecture (Conjecture 3.3) is equivalent to the union-closed sets conjecture (Conjecture 1.1).*

Proof. Suppose the union-closed sets conjecture (Conjecture 1.1) is true. Let G be a finite bipartite graph with at least one edge and with the vertex set $U\dot{\cup}W$ where $U = \sum_{w\in W} \mathrm{N(w)}$ and $W = \{w_1, w_2, \dots, w_n\}$ has no duplicate vertices. Let

$$\mathcal{A} = \{\mathrm{N(w_1)}, \mathrm{N(w_2)}, \dots, \mathrm{N(w_n)}\}.$$

Since W has no duplicate vertices, $\mathcal{A}$ has distinct elements and consequently $\mathcal{A}$ is a finite set. Since G has at least one edge, W contains a vertex with a nonempty neighborhood and consequently $\mathcal{A} \neq \{\varnothing\}$. Also $\mathcal{A}$ is a finite set of union-closed finite sets because for any $w_i, w_j \in W$ there exists $w_k \in W$ such that $\mathrm{N(w_i)} \cup \mathrm{N(w_j)} = \mathrm{N(w_k)}$. Then by the union-closed sets conjecture (Conjecture 1.1), there is an element $u \in \cup_{i=1}^{n} \mathrm{N(w_i)}$ that belongs to at least half of the sets in $\mathcal{A} = \{\mathrm{N(w_1)}, \mathrm{N(w_2)}, \dots, \mathrm{N(w_n)}\}$. Therefore there exists a vertex $u \in U$ of degree at least $\frac{1}{2}|\mathcal{A}| = \frac{1}{2}|W|$. Thus the union-closed graphs conjecture is true.

Conversely suppose the union-closed graphs conjecture (Conjecture 3.3) is true. Let $\mathcal{A} = \{S_1, S_2, \dots, S_n\}$ be a finite set of finite sets. Suppose $\mathcal{A} \neq \{\varnothing\}$ and $\mathcal{A}$ is union-closed.

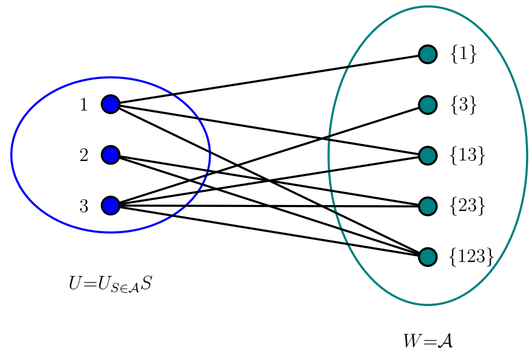

Figure 2: A union-closed bipartite graph G

Without loss of generality, let $S_n = \varnothing$. Let

$$U = \bigcup_{i=1}^{n} S_i = \{x_1, x_2, \dots, x_m\} \text{ and } W = \mathcal{A},$$

for some integer m. Define a bipartite graph G on $m+n$ vertices with the vertex set $V = U \dot\cup W$ where x_i is adjacent to S_j if $x_i \in S_j$. Note that $\mathrm{N(S_i)} = \mathrm{S_i}$ for all $i = 1, 2, \dots, n$. Then $U = \cup_{i=1}^{n} S_i = \cup_{i=1}^{n} \mathrm{N(S_i)}$. Since the set $\mathcal{A} \neq \{\varnothing\}$, $W = \mathcal{A}$ contains a vertex with a nonempty neighborhood and consequently G has an edge. Since $W = \mathcal{A}$ is a set, $\mathrm{N(S_i)} = \mathrm{S_i} \neq \mathrm{S_j} = \mathrm{N(S_j)}$ for all $i \neq j$. Thus W has no duplicate vertices. Since $\mathcal{A}$ is union-closed, for any $S_i, S_j \in \mathcal{A}$ there exists $S_k \in \mathcal{A}$ such that $S_i \cup S_j = S_k$ which implies $\mathrm{N(S_i)} \cup \mathrm{N(S_j)} = \mathrm{N(S_k)}$. Therefore G is a union-closed bipartite graph. By the union-closed graphs conjecture (Conjecture 3.3), there exists a vertex $x \in U = \cup_{i=1}^{n} S_i$ of degree at least $\frac{1}{2}|W| = \frac{1}{2}|\mathcal{A}|$. Therefore $x \in \cup_{A\in\mathcal{A}} A$ belongs to at least half of the sets in $\mathcal{A}$. Thus the union-closed sets conjecture is true. □

Example 3.5. To illustrate the above proof, consider the following union-closed finite set of finite sets:

$$\mathcal{A} = \{\{1\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

Suppose $W = \mathcal{A}$ and $U = \cup_{S\in\mathcal{A}} S = \{1,2,3\}$. Define a bipartite graph G with the vertex set $V = U \dot\cup W$ where $i \in U$ and $S_j \in \mathcal{A}$ are adjacent if $i \in S_j$. See Figure 2. We can verify that G is a union-closed bipartite graph with the vertex set $U \dot\cup W$. Also W has no duplicate vertices and vertices $1, 3 \in U$ have degrees at least $\frac{1}{2}|W| = 2.5$.

Now we present a hypergraph formulation of the UCSC. Although there is a natural connection between hypergraphs and bipartite graphs, we do not use any such connection directly in this hypergraph formulation and proving its equvalence to the UCSC. We introduce a special property of a hypergraph as follows:

Definition 3.6. A hypergraph $H = (V, E)$ is *union-closed* if for any two edges $e_i, e_j \in E$, there exists an edge $e_k \in E$ such that $e_i \cup e_j = e_k$.

Usually edges of a hypergraph $H = (V, E)$ are non-empty subsets of V. In this article we allow an edge to be the empty set and we call it the *empty-edge*.

Conjecture 3.7 (The union-closed hypergraphs conjecture)**.** Suppose a finite hypergraph $H = (V, E)$ with at least one edge which is not the empty-edge has no repeated edges and no isolated vertices (i.e., $V = \cup_{e \in E} e$). If $H = (V, E)$ is union-closed, then H has a vertex of degree at least $\frac{1}{2}|E|$.

Theorem 3.8. *The union-closed hypergraphs conjecture (Conjecture 3.7) is equivalent to the union-closed sets conjecture (Conjecture 1.1).*

Proof. Suppose the UCSC is true. Suppose a finite hypergraph $H = (V, E)$ with at least one edge which is not the empty-edge has no repeated edges and no isolated vertices (i.e., $V = \cup_{e \in E} e$). Then E is a finite set of finite sets and $E \neq \{\varnothing\}$. Suppose $H = (V, E)$ is union-closed. Then E is union-closed. By the UCSC, there is an element in $V = \cup_{e \in E} e$ that belongs to at least half of the sets in E, i.e., there is a vertex in $V = \cup_{e \in E} e$ of degree at least $\frac{1}{2}|E|$. Thus the union-closed hypergraphs conjecture is true.

Conversely suppose the union-closed hypergraphs conjecture is true. Let $\mathcal{A} = \{S_1, S_2, \ldots, S_n\}$ be a finite set of finite sets and $\mathcal{A} \neq \{\varnothing\}$. Without loss of generality, let $S_n = \varnothing$. Construct a finite hypergraph $H = (V, E)$ where $E = \mathcal{A} = \{S_1, S_2, \ldots, S_n\}$ and $V = \bigcup_{i=1}^{n} S_i$. Then H is a finite hypergraph with at least one edge which is not the empty-edge and H has no repeated edges and no isolated vertices. Suppose $\mathcal{A} = E$ is union-closed. Then $H = (V, E)$ is union-closed. By the union-closed hypergraphs conjecture, there is a vertex in $V = \bigcup_{i=1}^{n} S_i$ of degree at least $\frac{1}{2}|E| = \frac{1}{2}|\mathcal{A}|$, i.e., there is an element in $V = \bigcup_{i=1}^{n} S_i$ that belongs to at least half of the sets in $E = \mathcal{A}$. Thus the UCSC is true. □

From Theorems 3.8 and 2.5, we have the following result:

Corollary 3.9. *The union-closed hypergraphs conjecture (Conjecture 3.7) is equivalent to the addition-closed columns conjecture (Conjecture 2.4)*

Note that we can have an independent proof of the preceding corollary by using the fact that the columns of the incidence matrix of a union-closed hypergraph are addition-closed.

4 Future research

With three conjectures in this article equivalent to the UCSC, we can explore existing tools from linear algebra and graph theory to prove or disprove the UCSC. However properties of a union-closed bipartite graph or hypergraph can be independently studied. Similarly we can investigate linear algebraic properties such as rank, null space, and eigenvectors of a binary matrix whose columns are addition-closed.

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