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On outer-convex Roman dominating function in graphs

Leomarich F. Casinillo

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Abstract

Let $G = (V(G), E(G))$ be a connected graph and let $\phi : V(G) \rightarrow \{0, 1, 2\}$ be a Roman dominating function (RDF) on G. For each $j \in \{0, 1, 2\}$, let $V_j = \{x \in V(G) : \phi(x) = j\}$. Then ϕ can be represented as $\phi = (V_0, V_1, V_2)$. A function ϕ is an *outer-convex Roman dominating function* (OConRDF) on G if for each $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$ and V_0 is a convex set in G. The *weight* of OConRDF ϕ is denoted by $\widetilde{\omega}_G^{conR}(\phi)$ and is defined as $\widetilde{\omega}_G^{conR}(\phi) = \sum_{x\in V(G)} \phi(x)$, that is, $\widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2|$. Additionally, the *outer-convex Roman domination number* of G is denoted by $\widetilde{\gamma}_{conR}(G)$ and is defined as the minimum weight of an OConRDF on G, that is, $\widetilde{\gamma}_{conR}(G) = min\{\widetilde{\omega}_G^{conR}(\phi) : \phi \textit{ is an } \text{OConRDF} \textit{ on } G\}$. Furthermore, any OConRDF ϕ on G with $\widetilde{\omega}_G^{conR}(\phi) = \widetilde{\gamma}_{conR}(G)$ is called a $\widetilde{\gamma}_{conR}$-function on G. This paper introduces a new parameter of a Roman dominating function in graphs and discusses some theoretical properties, including bounds, the realization problem, and characterizations.

1 Introduction

Notably, domination in graphs has become a center of research interest for many discrete mathematicians, and there are now numerous papers in the literature regarding the topic [1, 4, 6, 8, 11, 12]. One of the motivating topics in domination in graphs is the Roman dominating function, which was initiated by Cockayne et al. [7] and is based on the defense strategy of Roman Emperor Constantine the Great. Some papers on Roman domination can be found in [2, 6, 12]. Meanwhile, in the year 2004, Lemanska [11] introduced the concept of convex domination number in graphs. On the other hand, the concept of outer-convex domination in graphs was pioneered by Dayap and Enriquez [8] in the year 2020. In that case, the author is inspired to link the concept of outer-convex domination and Roman

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domination as a new variant of the Roman dominating function in graphs. For the basic concepts and terminologies, the readers may refer to [3, 5].

Let $G = (V(G), E(G))$ be a simple and connected graph where $V(G)$ and $E(G)$ are the vertex and edge sets, respectively. A *walk* in a graph G is a finite sequence of alternating vertices and edges, which starts and ends with a vertex. A *path* is a walk of order n and size $n-1$, denoted by P_n. Let $u, v \in V(G)$ such that $u \neq v$. The *distance* between u and v is the length of the shortest path between u and v, and is denoted by $d_G(u, v)$. If there is no such path, then the distance is defined as $d_G(u, v) = \infty$. A $u - v$ path with length $d_G(u, v)$ is called $u - v$ geodesic. A closed interval $I_G[u, v]$ is defined as a set that consists of all vertices that lie on a $u - v$ geodesic in graph G. Let $W \subset V(G)$. The union of all sets $I_G[u, v]$ where $u, v \in W$ is denoted by $I_G[W]$, that is, $\cup_{u,v \in W} I_G[u, v] = I_G[W]$. Then $x \in I_G[W]$ if and only if x is in some $u - v$ geodesic for any $u, v \in W$. Hence, a set W is a convex set in G if and only if $I_G[W] = W$. Clearly, $V(G)$ is a convex set if G is connected. Let C be a convex set of G. Then, the convexity number of a connected graph G denoted by $con(G)$ is defined as the maximum cardinality of a convex set of vertices which does not contain all vertices of G, that is, $con(G) = max\{|C| : C \text{ is a convex set of } G\} < |V(G)|$ [9].

For the concepts and definitions used in this study that are not presented here, they can be found in [3, 5]. Let D be a subset of $V(G)$. Then D is called a *dominating set* of G if for each $v \in V(G) \setminus D$, there exists $u \in D$ such that $uv \in E(G)$, that is $N[D] = V(G)$ [10]. The *domination number* of G is denoted by $\gamma(G)$ and is defined as the smallest cardinality of a dominating set D in G. The dominating set with $|D| = \gamma(G)$ is called γ-set on G. Let C be a subset of $V(G)$. Then C is a *convex dominating set* on G if for each $v \in V(G) \setminus C$, there exists $u \in S$ such that $uv \in E(G)$ and C is a convex set on G [11]. The minimum cardinality of C is denoted by $\gamma_{con}(G)$ and is called the *convex domination number*. A convex dominating set C that satisfies $|C| = \gamma_{con}(G)$ is called γ_{con}-set on G. Let O be a subset of $V(G)$. Then O is called an *outer-convex dominating set* on G if for each $v \in V(G) \setminus O$, there exists $u \in O$ such that $uv \in E(G)$ and $V(G) \setminus O$ is a convex set on G [8]. The smallest cardinality of O is denoted by $\widetilde{\gamma}_{con}(G)$ and is called the *outer-convex domination number*. An outer-convex dominating set O that satisfies $|O| = \widetilde{\gamma}_{con}(G)$ is called $\widetilde{\gamma}_{con}$-set on G.

Let $\phi : V(G) \to \{0, 1, 2\}$. For each $j \in \{0, 1, 2\}$, let $V_j = \{x \in V(G) : \phi(x) = j\}$. Then $\phi = (V_0, V_1, V_2)$ is a function on G. A function $\phi : V(G) \to \{0, 1, 2\}$ is called a *Roman dominating function* (RDF) on G provided that for every vertex $x \in V_0$ there exists $y \in V_2$ such that $xy \in E(G)$ [7]. The *weight* of an RDF ϕ on G is defined by $\omega_G^R(\phi) = \sum_{v \in V(G)} \phi(v) = |V_1| + 2|V_2|$. The *Roman domination number* of G is denoted by $\gamma_R(G)$ and is defined as the smallest weight of an RDF ϕ on G, that is, $\gamma_R(G) = min\{\omega_G^R(\phi) : \phi \textit{ is an } \text{RDF} \textit{ on } G\}$. Additionally, every RDF ϕ on G with $\omega_G^R(\phi) = \gamma_R(G)$ is called a γ_R-function on G. A function ϕ is an *outer-convex Roman dominating function* (OConRDF) on G if for each $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$ and V_0 is a convex set in G. The *weight* of OConRDF ϕ is denoted by $\widetilde{\omega}_G^{conR}(\phi)$ and is defined as $\widetilde{\omega}_G^{conR}(\phi) = \sum_{x \in V(G)} \phi(x)$, that is, $\widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2|$. Additionally, the *outer-convex Roman domination number* of G is denoted by $\widetilde{\gamma}_{conR}(G)$ and is defined as the minimum weight of an OConRDF on G, that is, $\widetilde{\gamma}_{conR}(G) = min\{\widetilde{\omega}_G^{conR}(\phi) : \phi \textit{ is an } \text{OConRDF} \textit{ on } G\}$. Furthermore, any OConRDF ϕ on G with $\widetilde{\omega}_G^{conR}(\phi) = \widetilde{\gamma}_{conR}(G)$ is so called a $\widetilde{\gamma}_{conR}$-function on G. This paper presents a new variant of the Roman dominating function in graphs, namely outer-convex Roman domination. Some important theoretical properties of an outer-convex Roman dominating function were presented, including characterizations. Moreover, some bounds of outer-convex Roman domination number were presented, and a realization problem was constructed.

2 Results

This section presents some important theoretical properties of outer-convex Roman dominating function (OConRDF) on any simple and connected graph.

Proposition 2.1. Let G be a connected graph. If $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on G, then $V_1 \cup V_2$ is an outer-convex dominating set on G.

Proof. Assume that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on G. It implies that ϕ is a OConRDF on G and so, for each $v \in V_0$, there exists $u \in V_2$ such that $uv \in E(G)$, that is, $V_0 \subseteq N_G[V_2]$. In addition, by definition of OConRDF on G, it means that $V_0 = V(G) \setminus (V_1 \cup V_2)$ is a convex set on G. Therefore, it suffices to conclude that $V_1 \cup V_2$ is an outer-convex dominating set on G. □

Note that if $\phi = (V_0, V_1, V_2)$ is an OConRDF on G and $|V_0| = |V_2|$, then it is clear that $\widetilde{\gamma}_{conR}(G) = n$. In that case, the following result is immediate.

Proposition 2.2. Let G be a connected graph of order n and let $\phi = (V_0, V_1, V_2)$ be $\widetilde{\gamma}_{conR}$-function on G. Then $\widetilde{\gamma}_{conR}(G) < n$ if and only if $|V_2| < |V_0|$.

Proof. Let G be a simple and connected graph with $|V(G)| = n$ and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on G.
($\Rightarrow$) Assume that $\widetilde{\gamma}_{conR}(G) < n$. Then it is easy to check that $|V_0| \neq |V_2|$. Suppose that if $|V_2| > |V_0|$, then we have $\widetilde{\gamma}_{conR}(G) = |V_1| + 2|V_2| > |V_1| + |V_2| + |V_0| = |V(G)| = n$. This is a contradiction. Therefore, it is concluded that $|V_2| < |V_0|$.
($\Leftarrow$) Conversely, we assume that $|V_2| < |V_0|$. Then we have $\widetilde{\gamma}_{conR}(G) = |V_1| + 2|V_2| < |V_1| + |V_2| + |V_0| = |V(G)| = n$. □

Proposition 2.3. Let G be a connected graph and let $\phi = (V_0, V_1, V_2)$ be an OConRDF on G and $V_1 = \varnothing$. Then ϕ is $\widetilde{\gamma}_{conR}$-function on G if and only if V_2 is a $\widetilde{\gamma}_{con}$-set on G.

Proof. Let $\phi = (V_0, V_1, V_2)$ be an OConRDF on a simple and connected graph G.
($\Rightarrow$) Suppose that $V_1 = \varnothing$ and $\phi = (V_0, V_1, V_2)$ is $\widetilde{\gamma}_{conR}$-function on G. Seeking a contradiction. Assume for a moment that V_2 is not a $\widetilde{\gamma}_{con}$-set on G. Then there exists $\widetilde{\gamma}_{con}$-set D on G such that $V(G) \setminus D$ is a convex set. Set $W_0 = V(G) \setminus D$, $W_1 = \varnothing$, and $W_2 = D$. In that case, $\lambda = (W_0, W_1, W_2)$ is an OConRDF on G. Observe that $\widetilde{\omega}_G^{conR}(\lambda) = |W_1| + 2|W_2| = 2|W_2| < 2|V_2| = |V_1| + 2|V_2| = \widetilde{\gamma}_{conR}(G)$. Hence, we get $\widetilde{\omega}_G^{conR}(\lambda) < \widetilde{\gamma}_{conR}(G)$, a contradiction. Thus, it implies that V_2 is a $\widetilde{\gamma}_{con}$-set on G.
($\Leftarrow$) Suppose that V_2 is a $\widetilde{\gamma}_{con}$-set on G. Then it follows that $V_0 = V(G) \setminus V_2$ is a convex set in G. Seeking a contradiction. Assume for a moment that $\phi = (V_0, V_1, V_2)$ is an OConRDF on G but not a $\widetilde{\gamma}_{conR}$-function. And so, there exists $\widetilde{\gamma}_{conR}$-function $\psi = (U_0, U_1, U_2)$ on G for

which $U_1 = \varnothing$. Note that $\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\psi) = |U_1|+2|U_2| = 2|U_2| < 2|V_2| = |V_1|+2|V_2| = \widetilde{\omega}_G^{conR}(\phi)$. So, we have $|U_2| < |V_2|$, a contradiction to our assumption that V_2 is a $\widetilde{\gamma}_{con}$-set on G. Therefore, $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on G. □

If $|V_2| = 1$ and $|V_1| = 0$, then it is clear that V_2 is a $\widetilde{\gamma}_{con}$-set on G and in view of Proposition 2.3, $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on G. The next result shows the lower and upper bounds of outer-convex Roman domination number for any connected graph G with $|V(G)| = n \geq 1$.

Theorem 2.4. *Let G be a connected graph of order n. Then,*

$$max\{\gamma_R(G), \widetilde{\gamma}_{con}(G)\} \leq \widetilde{\gamma}_{conR}(G) \leq min\{2\widetilde{\gamma}_{con}(G), n\}.$$

Proof. Let G be a connected graph of order n and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on G. Invoking Proposition 2.1, it follows that $V_1 \cup V_2$ is an outer-convex dominating set on G. Note that $\widetilde{\gamma}_{con}(G) \leq |V_1| + |V_2| \leq |V_1| + 2|V_2| = \widetilde{\gamma}_{conR}(G)$ and so, we obtain $\widetilde{\gamma}_{con}(G) \leq \widetilde{\gamma}_{conR}(G)$. Since every OConRDF on a connected graph G is an RDF on G, it means that $\gamma_R(G) \leq \widetilde{\gamma}_{conR}(G)$. Accordingly, we get $max\{\widetilde{\gamma}_{con}(G), \gamma_R(G)\} \leq \widetilde{\gamma}_{conR}(G)$. Meanwhile, let $V_0 = \varnothing$. Then we have $V_2 = \varnothing$. In that case, the dominating function $\phi = (\varnothing, V_1 = V(G), \varnothing)$ is an OConRDF on G. And it follows that $\widetilde{\gamma}_{conR}(G) \leq \widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2| = |V_1| + 2(0) = |V_1| = |V(G)| = n$. So, we have $\widetilde{\gamma}_{conR}(G) \leq n$. Suppose that $\phi = (V_0, V_1, V_2)$ is a OConRDF on G for which $V_1 = \varnothing$. If ϕ is a γ_{conR}-function on G, then invoking Proposition 2.3, it means that V_2 is a γ_{con}-set on G and so, $V_0 = V(G) \setminus V_2$ is a convex set. Hence, it implies that $\widetilde{\gamma}_{conR}(G) = |V_1| + 2|V_2| = 0 + 2|V_2| = 2\widetilde{\gamma}_{con}(G)$. Consequently, we obtain $\widetilde{\gamma}_{conR}(G) \leq min\{n, 2\widetilde{\gamma}_{con}(G)\}$. Therefore, $max\{\gamma_R(G), \widetilde{\gamma}_{con}(G)\} \leq \widetilde{\gamma}_{conR}(G) \leq min\{2\widetilde{\gamma}_{con}(G), n\}$. □

Proposition 2.5. Let G be a nontrivial connected graph and $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on G. If $|V_2| = 1$, then for any $x, y \in V_0$, $d_{\langle V_0 \rangle}(x, y) \leq 2$.

Proof. Assume that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on any nontrivial connected graph G for which $|V_2| = 1$. Then it follows that $V_0 \neq \varnothing$. Let $x, y \in V_0$. If $x = y$, then $d_{\langle V_0 \rangle}(x, y) = 0 = d_G(x, y)$. Suppose $x \neq y$. Now, if $xy \in E(G)$, then we have $d_{\langle V_0 \rangle}(x, y) = 1 = d_G(x, y)$. Let $xy \notin E(G)$ and let $u \in V_2$. Note that $\{x, y\} \subseteq N_G[u]$. It implies that $xu, uy \in E(G)$ and so, $d_G(x, y) = 2$. Since ϕ is $\widetilde{\gamma}_{conR}$-function on G, it implies that V_0 is a convex set on G. Hence, $d_{\langle V_0 \rangle}(x, y) = 2 = d_G(x, y)$. Therefore, for any $x, y \in V_0$, $d_{\langle V_0 \rangle}(x, y) \leq 2$ whenever $|V_2| = 1$. □

The converse of Proposition 2.5 is not always true. For instance, let $P_4 = [v_1, v_2, v_3, v_4]$. By Theorem 2.4, $\widetilde{\gamma}_{conR}(P_4) \leq 4$, it means that we can set $V_0 = \{v_2, v_3\}$, $V_1 = \varnothing$, and $V_2 = \{v_1, v_4\}$. In that case, V_0 is a convex set in G and $d_{P_4}(v_2, v_3) = 1$ but $|V_2| \neq 1$.

Theorem 2.6. *Let G be a connected graph and $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on G. Then $|V_2| = 1$ and $V_1 = \varnothing$ if and only if $G = K_1 + H$ where $H \in \{K_{n\geq 1}, C_{4\leq n\leq 5}, P_3\}$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on any nontrivial connected graph G.
($\Rightarrow$) Assume that $|V_2| = 1$ and $V_1 = \varnothing$. Then it implies that $|V_0| \geq 1$. By Proposition 2.5, we have that for all $x, y \in V_0$, $d_G(x, y) \leq 2$. Now, let $G = K_1 + H$ where H is a connected graph. Since $|V_2| = 1$, we can let $V_2 = \{u\} = V(K_1)$. Suppose that $d_{\langle V_0 \rangle}(x, y) = 0$, that is, $x = y$. It follows that $G = K_1 + K_1$. Suppose $x \neq y$ and $d_{\langle V_0 \rangle}(x, y) = 1$ for all $x, y \in V_0$. In that case, we obtain $G = K_1 + K_{n \geq 2}$. Since it is clear that $V_0 = V(K_n)$ is a convex set in G, it follows that $H = K_n$ for all $n \geq 1$. Moreover, suppose that if $x \neq y$ and $1 \leq d_{\langle V_0 \rangle}(x, y) \leq 2$, then it implies that $G = K_1 + C_{4 \leq n \leq 5}$ or $G = K_1 + P_3$. And it is clear that $I_G[V(C_{4 \leq n \leq 5})] = V(C_{4 \leq n \leq 5})$ and $I_G[V(P_3)] = V(P_3)$. Thus, $H = C_{4 \leq n \leq 5}$ or $H = P_3$. Therefore, it suffices to conclude that $G = K_1 + H$ where $H \in \{K_{n \geq 1}, C_{4 \leq n \leq 5}, P_3\}$.
($\Leftarrow$) Assume that $G = K_1 + H$ where $H \in \{K_{n \geq 1}, C_{4 \leq n \leq 5}, P_3\}$. Then consider the following cases:
Case 1. $H = K_n$ for all $n \geq 1$
In this case, $G = K_1 + K_n = K_{n+1}$. Let $V_2 = \{u\} \subset V(G)$, and set $V_0 = V(G) \setminus V_2$ and $V_1 = \varnothing$. Then it follows that $\phi = (V(G) \setminus V_2, \varnothing, V_2)$ is an RDF on G. Since G is a complete graph, $\langle V_0 \rangle = \langle V(G) \setminus V_2 \rangle$ is also a complete graph and hence, V_0 is a convex set in G. Consequently, ϕ is an OConRDF on G and we obtain $|V_2| = 1$ and $V_1 = \varnothing$.
Case 2. $H = C_{4 \leq n \leq 5}$
It follows that $G = K_1 + C_{4 \leq n \leq 5}$. Let $V_2 = \{u\} = V(K_1)$. Then it implies that $V(C_{4 \leq n \leq 5}) \subseteq N_G[u]$. So, $\phi = (V(C_{4 \leq n \leq 5}), \varnothing, \{u\})$ is an RDF on G. Now, let $x, y \in V(C_{4 \leq n \leq 5})$. Then we obtain $1 \leq d_{C_{4 \leq n \leq 5}}(x, y) \leq 2$. since $xu, uy \in E(G)$, it follows that $I_G[V(C_{4 \leq n \leq 5})] = V(C_{4 \leq n \leq 5})$ and so, $V_0 = V(C_{4 \leq n \leq 5})$ is a convex set on G. Accordingly, ϕ is an OConRDF on G and we have $|V_2| = 1$ and $V_1 = \varnothing$.
Case 3. $H = P_3$
In this case, we get $G = K_1 + P_3$. Let $V_2 = \{u\} = V(K_1)$. Then, we set $V_0 = V(G) \setminus V_2 = V(P_3)$ and $V_1 = \varnothing$. And so, $\phi = (V_0, V_1, V_2)$ is an RDF on G. Using the same argument as in Case 2, we also have $I_G[V(P_3)] = V(P_3)$ and hence, ϕ is an OConRDF on G. Therefore, we end up with $|V_2| = 1$ and $V_1 = \varnothing$. $\square$

The following corollary is an immediate consequence of Proposition 2.3 and Theorem 2.6.

Corollary 2.7. *Let $G \neq K_1 + H$ where $H \in \{K_{n \geq 1}, C_{4 \leq n \leq 5}, P_3\}$ be a connected graph of order $n \geq 4$ and let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on G. Then $|V_2| = 2$ and $V_1 = \varnothing$ if and only if there exists a dominating set $\{x, y\}$ such that $V(G) \setminus \{x, y\}$ is a convex set.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on a connected graph $G \neq K_1 + H$ where $H \in \{K_{n \geq 1}, C_{4 \leq n \leq 5}, P_3\}$.
($\Rightarrow$) Assume that $|V_2| = 2$ and $V_1 = \varnothing$. In view of Proposition 2.3, it follows that V_2 is a $\widetilde{\gamma}_{con}$-set on G. This implies that V_2 is an outer-convex dominating set on G and so, V_0 is a convex set in G. Since $|V_2| = 2$, we let $V_2 = \{x, y\}$. Therefore, it suffices to say that $\{x, y\}$ is a dominating set and $V_0 = V(G) \setminus \{x, y\}$ is a convex set in G.
($\Leftarrow$) Assume that there exists a dominating set $\{x, y\}$ such that $V(G) \setminus \{x, y\}$ is a convex set in G. Then it implies that $D = \{x, y\}$ is an outer-convex dominating set on G. Since $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on G and $V_1 = \varnothing$, it follows that V_2 is a $\widetilde{\gamma}_{con}$-set on G by Proposition 2.3. Hence, $2 = |D| \geq |V_2|$. Now if $|V_2| = 1$, then by Theorem 2.6, we

get $G = K_1 + H$ where $H \in \{K_{n\geq 1}, C_{4\leq n\leq 5}, P_3\}$, contrary to our assumption. So, we have $|V_2| \geq 2$. Consequently, we end up with $|V_2| = 2$ and $V_1 = \varnothing$. □

Theorem 2.8. *Let G be a connected graph with $|V(G)| = n$ and let $\phi = (V_0, V_1, V_2)$ be an OConRDF on G for which $V_1 = \varnothing$. Then ϕ is a $\widetilde{\gamma}_{conR}$-function on G and $\widetilde{\gamma}_{conR}(G) = \widetilde{\gamma}_{con}(G) + 1$ if and only if there exists $v \in V(G)$ such that $deg_G(v) = n - \widetilde{\gamma}_{con}(G)$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a OConRDF on a connected graph G of order n for which $V_1 = \varnothing$.
($\Rightarrow$) Assume that ϕ is a $\widetilde{\gamma}_{conR}$-function on G and $\widetilde{\gamma}_{conR}(G) = \widetilde{\gamma}_{con}(G) + 1$. Then it implies that $\widetilde{\gamma}_{con}(G) + 1 = |V_1| + 2|V_2|$. Since $|V_1| = 0$, by Proposition 2.3, it follows that V_2 is a $\widetilde{\gamma}_{con}$-set on G and so, $\widetilde{\gamma}_{con}(G) = |V_2|$. So, we get $|V_2| + 1 = 2|V_2|$ and it means that $|V_2| = 1$. Let $V_2 = \{v\} \subseteq V(G)$. Since V_2 is a $\widetilde{\gamma}_{con}$-set on G, it suffices to say that $V(G) = N_G[v]$ and $V(G) \setminus \{v\}$ is a convex set. This follows that $deg_G(v) = N_G(v) = |V(G)| - |\{v\}| = n - |V_2| = n - \widetilde{\gamma}_{con}(G)$.
($\Leftarrow$) Assume that there exists $v \in V(G)$ such that $deg_G(v) = n - \widetilde{\gamma}_{con}(G)$. Let $V_2 = \{v\}$. Since $|V_1| = 0$, it is clear that $\widetilde{\gamma}_{con}(G) = |V_2|$ and V_2 is a outer-convex dominating set on G. In that case, $deg_G(v) = n - |V_2| = |V(G) \setminus V_2|$ and thus, $V(G) = N_G[V_2]$. Since V_2 is an outer-convex dominating set on G, it implies that $V(G) \setminus V_2$ is a convex set. Then we set $V_0 = V(G) \setminus V_2$, $V_1 = \varnothing$. Since $|V_2|$ is a $\widetilde{\gamma}_{con}$-set on G, by Proposition 2.3, it follows that $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on G. To this end, we obtain $\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2| = |V_2| + |V_2| = \widetilde{\gamma}_{con}(G) + 1$. □

The next result is a realization problem.

Theorem 2.9. *Let a, b and n be positive integers such that $1 \leq a \leq b \leq n$. Then there exists a connected graph G such that $\gamma_R(G) = a$, $\widetilde{\gamma}_{conR}(G) = b$, and $|V(G)| = n$.*

Proof. Let $\lambda = (W_0, W_1, W_2)$ be a γ_R-function and $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on G. Then consider the following cases:
Case 1. $1 \leq a \leq b = n$
If $G = K_1$, then it is clear that $1 = \gamma_R(G) = \widetilde{\gamma}_{conR}(G) = |V(G)|$. Let $G = K_1 + \overline{K_{n-1}}$ where $n \geq 3$. Set $W_0 = V(\overline{K_{n-1}})$, $W_1 = \varnothing$ and $W_2 = V(K_1)$. Then it implies that $\gamma_R(G) = \omega_G^R(\lambda) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. Note that a trivial graph is a convex graph. Hence, we can set $V_0 = \{u\} \subseteq V(\overline{K_{n-1}})$, $V_1 = V(\overline{K_{n-1}}) \setminus \{u\}$, and $V_2 = V(K_1)$. So, it follows that $\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2| = |V(\overline{K_{n-1}}) \setminus \{u\}| + 2|V(K_1)| = (n-2) + 2(1) = n$. Hence, $1 < \gamma_R(G) < \widetilde{\gamma}_{conR}(G) = n$ and Case 1 holds.
Case 2. $1 < a < b < n$
Consider the graph $G = K_1 + P_{n-1}$ where $n \geq 5$. Then $|V(G)| = n \geq 5$. Set $W_0 = V(P_{n-1})$, $W_1 = \varnothing$ and $W_2 = V(K_1)$. So, we get $\gamma_R(G) = \omega_G^R(\lambda) = |W_1| + 2|W_2| = 2|V(K_1)| = 2$. Now, let $P_{n-1} = [v_1, v_2, ..., v_{n-1}]$. By Theorem 2.6, we can set $V_0 = \{v_1, v_2, v_3\}$, $V_1 = \{v_4, ..., v_{n-1}\}$ and $V_2 = V(K_1)$. So, $\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2| = |\{v_4, ..., v_{n-1}\}| + 2|V(K_1)| = (n-4) + 2(1) = n - 2$. Since $n \geq 5$, we therefore obtain $1 < \gamma_R(G) < \widetilde{\gamma}_{conR}(G) < n$ and Case 2 holds.
Case 3. $1 < a = b < n$
Let $G = K_n$ where $n \geq 3$. Let $v \in V(G)$. Then we set $W_0 = V(G) \setminus \{v\}$, $W_1 = \varnothing$ and $W_2 =$

$\{v\}$. Hence, $\gamma_R(G) = \omega_G^R(\lambda) = |W_1| + 2|W_2| = 2|\{v\}| = 2$. Note that $V(G) \setminus \{v\} = K_{n-1}$ and so, $V(K_{n-1})$ is a convex set in G. Thus, we let $V_i = W_i$ for all $i \in \{0, 1, 2\}$. And so, $\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2| = 2$. Therefore, $1 < \gamma_R(G) = \widetilde{\gamma}_{conR}(G) < n$ and Case 3 holds. □

The next corollary and a remark are consequences of Theorem 2.9.

Corollary 2.10. *Let G be a connected graph with $|V(G)| = n$. Then the difference $\widetilde{\gamma}_{conR}(G) - \gamma_R(G)$ can be made arbitrarily large.*

Proof. Let m be a positive integer. By Theorem 2.9, there exists a connected graph such that $\widetilde{\gamma}_{conR}(G) = m + 2 < n$ and $\gamma_R(G) = 2$. It follows that $\widetilde{\gamma}_{conR}(G) - \gamma_R(G) = m$. Since m is a positive integer, the conclusion follows. □

Remark 2.11. Let G be a connected graph with $|V(G)| = n$. Then the difference $n - \widetilde{\gamma}_{conR}(G)$ can be made arbitrarily large.

Theorem 2.12. *Let G be a connected graph. Then $\widetilde{\gamma}_{conR}(G) = 1$ if and only if $G = K_1$.*

Proof. Let $\phi = (V_0, V_1, V_2)$ be a $\widetilde{\gamma}_{conR}$-function on a connected graph G. Suppose $\widetilde{\gamma}_{conR}(G) = 1$. Assume for a moment that $G \neq K_1$. Then we have $|V(G)| > 1$. If $V_0 \neq \varnothing$, then it follows that $V_2 \neq \varnothing$. Thus, we obtain $\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\phi) = |V_1| + 2|V_2| > 1$, a contradiction. If $V_0 = \varnothing$, then we get $V_2 = \varnothing$. Hence, it implies that $\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\phi) = |V_1| = |V(G)| > 1$. This is also a contradiction. Therefore, we end up with $G = K_1$. The converse is clear. □

A convex number of G with respect to a subgraph H of G is denoted by $con_H(G)$ and is defined as the largest cardinality of a convex set of vertices in G that is a subset of $V(H)$.

Theorem 2.13. *Let $G = K_1 + H$ where H is a connected graph. Then the following holds:*

i.) $\widetilde{\gamma}_{conR}(G) = 2$ *if* $con_H(G) = |V(H)|$; *and*

ii.) $\widetilde{\gamma}_{conR}(G) = 2 + |V(H)| - con_H(G)$ *if* $con_H(G) < |V(H)|$.

Proof. Suppose $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}$-function on $G = K_1 + H$ where H is a connected graph. Then consider the following cases:
Case 1. $con_H(G) = |V(H)|$
In this case, $I_G[V(H)] = V(H)$ implying that $V(H)$ is a convex set of G. Let $V_2 = \{u\} = V(K_1)$. It implies that $V_0 = V(G) \setminus \{u\} = V(H)$ and $V_1 = \varnothing$. By Proposition 2.3, if follows that V_2 is a $\widetilde{\gamma}_{con}$-set on G, that is, $\widetilde{\gamma}_{con}(G) = |V_2|$. Invoking Theorem 2.8, we have $\widetilde{\gamma}_{conR}(G) = \widetilde{\gamma}_{con}(G) + 1 = |V_2| + 1 = 2$.
Case 2. $con_H(G) < |V(H)|$
This means that $I_G[V(H)] \neq V(H)$ and so, $V(H)$ is not a convex set in G. Let $v \in V(K_1)$. Since $G = K_1 + H$, it follows that $V(G) = N_G[v]$. Thus, we can set $V_2 = \{v\}$. Observe that $V(G) \setminus V_2 = V(H)$, implying that $V(G) \setminus V_2$ is not a convex set in G. In that case, we have $V_1 \neq \varnothing$. Let $M \subset V(H)$ be the largest convex set in G with respect to H, that is,

$con_H(G) = |M|$. Since $\phi = (V_0, V_1, V_2)$ is a $\widetilde{\gamma}_{conR}(G)$-function on G and M is a convex set in G, we can set $V_1 = V(H) \setminus M$ and $V_0 = M$. To this end, we obtain

$$\begin{aligned}\widetilde{\gamma}_{conR}(G) = \widetilde{\omega}_G^{conR}(\phi) &= |V_1| + 2|V_2| \\ &= |V(H) \setminus M| + 2|\{v\}| \\ &= 2 + |V(H)| - con_H(G).\end{aligned}$$

This completes the proof. □

Corollary 2.14. *Let n be a positive integer. Then the following are true:*

i.) $\widetilde{\gamma}_{conR}(K_n) = 2$ *where* $n \geq 3$*;*

ii.) $\widetilde{\gamma}_{conR}(S_n) = n + 1$ *where* $n \geq 1$*;*

iii.) $\widetilde{\gamma}_{conR}(W_n) = \begin{cases} 2, & \textit{if } n \in \{3, 4, 5\}, \\ 2 + |V(C_n)| - con_{C_n}(W_n), & \textit{if } n \geq 6, \end{cases}$

iv.) $\widetilde{\gamma}_{conR}(F_n) = \begin{cases} 2, & \textit{if } n \in \{2, 3\}, \\ 2 + |V(P_n)| - con_{P_n}(F_n), & \textit{if } n \geq 4. \end{cases}$

The next result is a characterization of an OConRDF in the join of two connected graphs.

Theorem 2.15. *Let G and H be connected graphs. Then $\phi = (V_0, V_1, V_2)$ is an OConRDF on $G + H$ if and only if one of the following is true:*

i.) $V_1 \cup V_2 = V_1^G \cup V_2^G$*; or*

ii.) $V_1 \cup V_2 = V_1^H \cup V_2^H$*; or*

iii.) $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$*,*

where $\phi|_G = (V_0^G, V_1^G, V_2^G)$ and $\phi|_H = (V_0^H, V_1^H, V_2^H)$ are OConRDF on G and H, respectively, such that $V_i^G = V_i \cap V(G)$ and $V_i^H = V_i \cap V(H)$, for each $i \in \{0, 1, 2\}$.

Proof. ($\Rightarrow$) Suppose $\phi = (V_0, V_1, V_2)$ is an OConRDF on $G + H$ where G and H are both connected graphs. Then we set $O_G = (V_1 \cup V_2) \cap V(G)$, $O_H = (V_1 \cup V_2) \cap V(H)$. Let $v \in V_0$. So, $v \in V(G) \setminus O_G$ or $v \in V(H) \setminus O_H$. Without loss of generality (WLOG), suppose $v \in V(G) \setminus O_G$. Since ϕ is an OconRDF on $G + H$, it implies that there exists $u \in V_2$ such that $uv \in E(G + H)$ and either $V_1 = V(G)$ or V_0 is a convex set on G. Then it follows that either $u \in O_G$ or $u \in O_H$. If $u \in O_G$, then it implies that O_G is an outer-convex dominating set on G. Let $V_0^G = V(G) \setminus O_G$, $V_1^G = O_G \setminus V_2$ and $V_2^G = O_G \setminus V_1$. Thus, $V_i^G = V_i \cap V(G)$ for each $i \in \{0, 1, 2\}$. Accordingly, $\phi|_G = (V_0^G, V_1^G, V_2^G)$ is an OConRDF on G. Since $V(G + H) = N_{G+H}[V_1^G \cup V_2^G]$ and $V_0^G \cup V(H)$ is a convex set in $G + H$, it follows that $V_1 \cup V_2 = V_1^G \cup V_2^G$ and so, $(i.)$ holds. On the other hand, let $V_0^H = V(H) \setminus O_H$, $V_1^H = O_H \setminus V_2$ and $V_2^H = O_H \setminus V_1$. Using a similar argument, it follows that $V_i^H = V_i \cap V(H)$ for any $i \in \{0, 1, 2\}$. Hence, $\phi|_H = (V_0^H, V_1^H, V_2^H)$ is an OConRDF on

H. Since $V(G+H) = N_{G+H}[V_1^H \cup V_2^H]$ and $V_0^H \cup V(G)$ is a convex set on $G+H$, it implies that $V_1 \cup V_2 = V_1^H \cup V_2^H$ and thus, $(ii.)$ holds. Consequently, if $u \in O_H$, then $V_0^G \cup V_0^H$ is a convex set on $G+H$ and so, we also have $V_1 \cup V_2 = (V_1^G \cup V_2^G) \cup (V_1^H \cup V_2^H)$. Thus, $(iii.)$ is satisfied.
$(\Leftarrow)$ Suppose either $(i.)$ or $(ii.)$ or $(iii.)$ is satisfied. Since $\phi|_G = (V_0^G, V_1^G, V_2^G)$ and $\phi|_H = (V_0^H, V_1^H, V_2^H)$ are OConRDF on G and H, respectively, it simply follows that $\phi = (V_0, V_1, V_2)$ is an OConRDF on $G+H$. □

The following results are a direct consequence of Theorem 2.15.

Corollary 2.16. *Let G and H be connected graphs. Then*

$$\widetilde{\gamma}_{conR}(G+H) = min\{\widetilde{\gamma}_{conR}(G), \widetilde{\gamma}_{conR}(H)\}.$$

Proof. Let $\alpha = (V_0^G, V_1^G, V_2^G)$ be a $\widetilde{\gamma}_{conR}$-function on G. Since $V(G+H) = N_{G+H}[V_1^G \cup V_2^G]$ and $V(G+H)\setminus(V_1^G \cup V_2^G)$ is a convex set on $G+H$, we can set $V_i = V_i^G$ for each $i \in \{0,1,2\}$. By Theorem 2.15, $\phi = (V_0, V_1, V_2)$ is an OConRDF on $G+H$. So, we obtain

$$\begin{aligned}\widetilde{\gamma}_{conR}(G+H) \leq \widetilde{\omega}_{G+H}^{conR}(\phi) &= |V_1| + 2|V_2| \\ &= |V_1^G| + 2|V_2^G| \\ &= \widetilde{\omega}_{G+H}^{conR}(\alpha) \\ &= \widetilde{\gamma}_{conR}(G).\end{aligned}$$

Let $\beta = (V_0^H, V_1^H, V_2^H)$ be a $\widetilde{\gamma}_{conR}$-function on H. Using similar argument, we get $\widetilde{\gamma}_{conR}(G+H) \leq \widetilde{\omega}_{G+H}^{conR}(\beta) = \widetilde{\gamma}_{conR}(H)$. Thus, $\widetilde{\gamma}_{conR}(G+H) \leq min\{\widetilde{\gamma}_{conR}(G), \widetilde{\gamma}_{conR}(H)\}$. Now, define $k = min\{\widetilde{\gamma}_{conR}(G), \widetilde{\gamma}_{conR}(H)\}$. Let $\phi' = (V_0', V_1', V_2')$ be a $\widetilde{\gamma}_{conR}$-function on $G+H$. By Theorem 2.15, $\phi|_G' = (V_0' \cap V(G), V_1' \cap V(G), V_2' \cap V(G))$ and $\phi|_H' = (V_0' \cap V(H), V_1' \cap V(H), V_2' \cap V(H))$ are OConRDF on G and H, respectively. Since $V(G+H) = N_{G+H}[V_1^G \cup V_2^G]$ and $V(G+H) \setminus (V_1^G \cup V_2^G)$ is a convex set on $G+H$, it follows that

$$\begin{aligned}\widetilde{\gamma}_{conR}(G+H) = \widetilde{\omega}_{G+H}^{conR}(\phi') &= |V_1'| + 2|V_2'| \\ &= |V_1' \cap V(G)| + 2|V_2' \cap V(G)| \\ &= \widetilde{\omega}_{G+H}^{conR}(\phi|_G') \\ &\geq \widetilde{\gamma}_{conR}(G) \\ &\geq k.\end{aligned}$$

By similar argument, we also get $\widetilde{\gamma}_{conR}(G+H) \geq \widetilde{\gamma}_{conR}(H) \geq k$. Consequently, $\widetilde{\gamma}_{conR}(G+H) \geq min\{\widetilde{\gamma}_{conR}(G), \widetilde{\gamma}_{conR}(H)\}$. Therefore, we end up with

$$\widetilde{\gamma}_{conR}(G+H) = min\{\widetilde{\gamma}_{conR}(G), \widetilde{\gamma}_{conR}(H)\}.$$

□

Remark 2.17. Let G and H be complete graphs. Then $\widetilde{\gamma}_{conR}(G+H) = 2$.

3 Conclusion

A new restricted variation of the Roman dominating function in graphs, called the outer-convex Roman dominating function (OConRDF), was introduced. Some important theoretical properties of OConRDF were presented and given elegant mathematical proofs. It is portrayed that for any connected graph G with $|V(G)| = n$, the lower bound of $\widetilde{\gamma}_{conR}(G)$ is equal to $max\{\gamma_R(G), \widetilde{\gamma}_{con}(G)\}$ and the upper bound is equal to $min\{2\widetilde{\gamma}_{con}(G), n\}$. The paper depicted that for positive integers a, b and n with $1 \leq a \leq b \leq n$, there exists a connected graph G with $|V(G)| = n$ such that $\gamma_R(G) = a$ and $\widetilde{\gamma}_{conR}(G) = b$. Hence, the difference $\widetilde{\gamma}_{conR}(G) - \gamma_R(G)$ can be made arbitrarily large. Moreover, the OConRDF is characterized in the join of two connected graphs. For future research, characterization of OConRDF under binary operations such as corona, Cartesian, strong, and lexicographic products is recommended to strengthen the current paper.

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Contact Information

Leomarich F. Casinillo
leomarichcasinillo02011990@gmail.com

Department of Mathematics
Visayas State University
Baybay City, Leyte, Philippines
https://orcid.org/0000-0003-3966-8836