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Very strongly connected graphs

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Abstract

A simple connected graph is strongly connected if there exists a directed path between every pair of vertices in both directions. Robbins showed that every 2-edge-connected graph can be given edge orientations that result in the graph being strongly connected. But a random assignment of edge directions may or may not result in a graph being strongly connected. We say that a 2-edge-connected graph is *very strongly connected* if any choice of edge orientations that does not feature a vertex having maximal or minimal indegree yields a strongly connected graph. We classify all graphs as either very strongly connected or not very strongly connected.

1 Introduction

A *graph* $G = G(V, E)$ consists of a set V of *vertices* (singular is *vertex*) and a set E of *edges* that connect some pairs of vertices. An edge is said to be *incident* to its endpoint vertices, and two vertices joined by an edge are said to be *adjacent*. A *path* in a graph is a sequence of edges that joins a sequence of distinct vertices. A graph is *connected* if there exists a path between every pair of vertices. A graph is *simple* if there are no “loop” edges from a vertex to itself and there are not multiple edges between pairs of vertices. A *directed graph* (or *digraph*) assigns a direction (or orientation) to each edge in the graph. In the rest of this paper, all graphs are assumed to be simple and connected.

The *degree* of a vertex is the number of edges that are incident to the vertex. The *indegree* of a vertex is the number of edges directed toward a vertex minus the number of edges directed away from the vertex. Note that a vertex has maximal indegree if every edge incident to the vertex is oriented toward the vertex (i.e., a sink vertex), and a vertex has minimal indegree if every edge is oriented away from the vertex (i.e., a source vertex).

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A graph is *strongly connected* if there exists a directed path between every pair of vertices in both directions [2]. It is easy to see that a graph with a source, sink, or bridge, or a graph that is not connected, cannot be strongly connected. It therefore seems natural to ask the following question: if a connected graph has no bridges, are there necessary and sufficient conditions that allow one to conclude that every choice of edge orientations that does not create sources or sinks (i.e., any valid orientation) result in the graph being strongly connected? This leads to the following characterization.

Definition 1.1. A graph G is *very strongly connected*, or *VSC*, when every valid orientation of G yields a strongly connected graph. A graph that is not VSC is said to be *anti-VSC*.

It is clear that every VSC graph is also strongly connected, but the converse is false. For example, consider the cube graph Q_3. There are edge orientations that result in a strongly connected graph, but Q_3 is anti-VSC because one can find a choice of edge orientations that results in a graph that is not strongly connected, as seen in Figure 1.

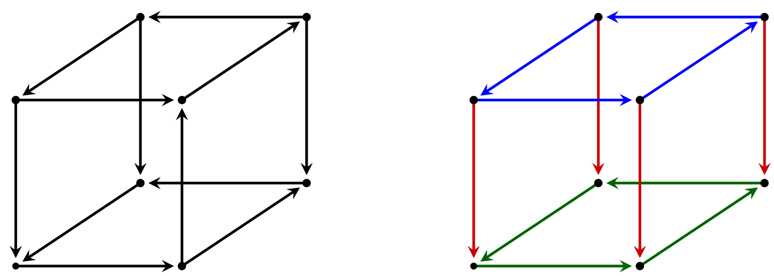

Figure 1: Strongly connected and anti-VSC cube graph edge orientations, respectively.

Since graphs with sources or sinks cannot be strongly connected, we will only consider graphs whose vertices do not have minimal or maximal indegree. We will also impose two other conditions on all graphs we analyze. First, since any graph vertex of degree 2 is an internal vertex of a directed path, without loss of generality all such directed paths can be thought of single directed edges. Second, we will only consider graphs that are *two-edge connected*, that is, graphs that remain connected if any edge is removed.

We note that a graph being VSC is related to how one analyzes certain positions in the combinatorial game Sylver Coinage. Finite game boards in Sylver Coinage can be represented by digraphs, and there is a connection between the digraph being VSC and the corresponding game position being a losing position for the next player. This realization was the catalyst for the authors to begin investigating variations of strongly connected graphs.

The remainder of this paper is structured as follows. In Section 2, motivated by a result of Lovász [1], we will characterize all finite simple connected bridgeless graphs that are VSC. In Section 3, we will prove that all other finite simple connected bridgeless graphs are anti-VSC.

2 Families of VSC graphs

There are many families of graphs one could consider when trying to determine if the given graphs are VSC or anti-VSC. In this section, motivated by the following result of Lovász

(1965), we will investigate four specific graph families [1].

Theorem 2.1 (Lovász)**.** *Let G be any finite graph whose vertices all have at least degree 3. Then G does not contain two vertex-disjoint cycles if and only if*

1. *For some vertex v of G, $G \setminus \{v\}$ is a forest, or*
2. *G is a wheel graph, or*
3. *G is a complete graph on 4 or 5 vertices, or*
4. *$G \setminus \{u, v, w\}$ is edgeless for some vertex triple u, v, w of G.*

We will now show that each of the four types of graphs identified in Theorem 2.1 is VSC. Moreover, in our context of simple connected bridgeless graphs whose vertices all have at least degree 3, we will soon conclude that these four types of graphs completely characterize all VSC graphs.

Theorem 2.2. *If G is a finite simple connected bridgeless graph whose vertices all have at least degree 3, and if for some vertex v of G, $G \setminus \{v\}$ is a forest, then G is VSC.*

Proof. Suppose G is a graph containing some vertex v for which $G \setminus \{v\}$ is a forest. To show that G is VSC, we must show that in any orientation of G which is source-free and sink-free, for any $u \neq v$, there exists a directed path from u to v and a directed path from v to u. We fix $u \neq v$ and construct such paths. First, for a path from u to v, consider a longest directed path in G which begins at u. Call this $uu_1u_2...u_i$. We claim that $v \in \{u, u_1, u_2, ..., u_i\}$. Indeed, suppose not. Since u_i is not a sink, there exists an edge u_iu_{i+1} directed away from u_i; since $v \notin \{u, u_1, u_2, ..., u_i\}$, the graph induced on $\{u, u_1, u_2, ..., u_i\}$ must be acyclic, and so $u_{i+1} \notin \{u, u_1, u_2, ..., u_i\}$. Thus, $uu_1u_2...u_iu_{i+1}$ is a strictly longer directed path starting at u, a contradiction. We conclude that $v \in \{u, u_1, u_2, ..., u_i\}$, so there exists a directed path from u to v.

A similar argument shows that there exists a directed path from v to u. It now follows that there exist paths between every pair of vertices of G in both directions, and hence G is VSC. □

Definition 2.3. Given any $n \in \mathbb{N}, n \geq 3$, the *wheel graph*, denoted W_n, is formed by a single *hub vertex* connected by radial edges, or *spokes*, to each of the n vertices of a cycle.

In order to prove that wheel graphs are VSC, we will need to define and analyze a new type of graph.

Definition 2.4. Given any $n \geq 3$, the *n-chord graph*, denoted ch_n, is a cycle formed from $n + 1$ vertices and having n chords that all share a common vertex - that is, ch_n has one vertex of degree n, denoted v_n, $n - 2$ vertices of degree 3, and 2 vertices of degree 2 that are adjacent to v_n.

See Figure 2 for an image of ch_6. We note that n-chord graphs feature two vertices of degree 2. This violates a restriction imposed in the last section on the types of graphs we will investigate, but it is a necessary (and temporary) means to an end.

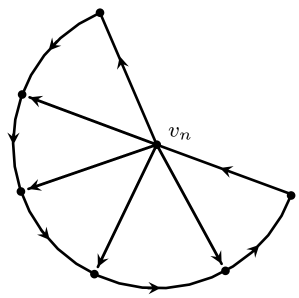

Figure 2: An oriented ch_6 whose outer cycle is a directed cycle.

Theorem 2.5. *For all $n \geq 3$, ch_n is VSC.*

Proof. It is easy to verify that ch_3, ch_4 and ch_5 are VSC for any valid orientations. Moreover, given any valid orientation of ch_n, if the outer cycle is a directed cycle, then clearly ch_n is strongly connected.

Now assume that the outer cycle of ch_n is not a directed cycle. Using induction, given $k \geq 5$, assume that each of $ch_3, ch_4, ..., ch_k$ are VSC, and consider ch_{k+1}. There must be at least one cycle vertex, say v', which is distinct from v_n, has degree 3, and is an endpoint to cycle edges with opposite orientations (i.e., both cycle edges are directed toward v' or both are directed away from v'). See Figure 3 for an example of one of these scenarios.

We will now partition ch_{k+1} into two subgraphs that share the chord $v'v_n$ as a common edge. There are now two cases to consider.

1. If v' is adjacent to a degree 2 vertex, one subgraph is a directed 3-cycle and the other is ch_k. Both subgraphs are source-free, sink-free, and strongly connected (the k-chord graph via the inductive hypothesis). This case is depicted in Figure 3.

2. If v' is not adjacent to a degree 2 vertex, by construction both subgraphs are in fact "smaller" n-chord graphs, say ch_s and ch_t, that are also source-free and sink-free. Since both subgraphs have valid orientations, by the inductive hypothesis they are VSC.

Thus, given any orientation of ch_{k+1}, the two corresponding subgraphs also have valid orientations, are strongly connected and share at least one vertex, and so ch_{k+1} is strongly connected. Consequently, ch_{k+1} is VSC, and via induction the result follows. □

Theorem 2.6. *For all $n \geq 3$, W_n is VSC.*

Proof. It is easy to see that W_3 is VSC. Given any W_n with $n \geq 4$ and any valid edge orientation, if the outer cycle is directed, it follows that W_n must be strongly connected.

If the outer cycle is not a directed cycle, the cycle must consist of at least two directed paths. This implies that there are at least two cycle vertices, say v_i and v_o, whose incident edges have opposite orientations: the corresponding spoke edges must be directed away from v_i and towards v_o. Hence, W_n can then be partitioned into 2 subgraphs that share these directed spokes. There are two cases to consider:

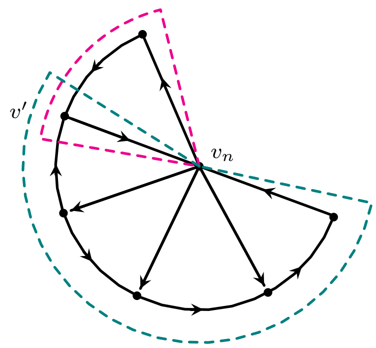

Figure 3: An oriented and subdivided ch_6 whose outer cycle is not a directed cycle.

1. If v_i and v_o are adjacent, one subgraph is a directed 3-cycle and the other is ch_n. Both subgraphs have valid orientations and are clearly strongly connected, and hence the wheel itself is strongly connected.

2. If v_i and v_o are not adjacent, by construction both subgraphs are "smaller" n-chord graphs, say ch_s and ch_t, that have valid orientations and are therefore strongly connected. Thus, the original wheel is also strongly connected. This case is depicted in Figure 4.

In both cases, the corresponding wheel graph is strongly connected. Thus, W_n is VSC. See Figure 4 for this argument applied to W_{12}. □

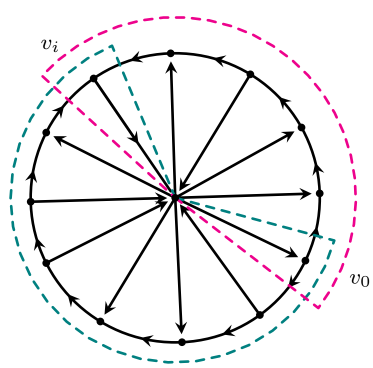

Figure 4: Subdividing W_{12} into overlapping 8-chord and 6-chord graphs.

For $n \geq 3$, we let K_n denote the family of *complete graphs* on n vertices. With our restriction to not allow graphs to have vertices with minimal or maximal indegree, it is easy to see that each of K_3, K_4, and K_5 are VSC (verification of this is left to the reader). But if $n \geq 6$, K_n is anti-VSC, which will be verified in the next section.

The final result of this section requires the following definitions.

Definition 2.7. The *join* of two disjoint graphs G and H, denoted $G + H$, is formed by connecting every vertex of G with every vertex of H.

Definition 2.8. Given any $n \in \mathbb{N}$, we let $J_n = C_3 + V_n$ denote the join of a 3-cycle C_3 with V_n, a set of n isolated vertices.

Given that G is a finite simple connected graph whose vertices all have at least degree 3, suppose that $G \setminus \{v, u, w\}$ is edgeless for some vertex triple v, u, w of G. Let $X = \{u, v, w\}$ and let Y denote the vertices in $G \setminus X$ with $|Y| = n$. Note that each vertex in Y has degree 3. The structure of G is completely determined by the subgraph H of G with $V(H) = X$ together with any edges connecting the vertices of H. There are 4 possibilities:

1. If H is edgeless, G is isomorphic to the complete bipartite graph $K_{3,n}$.
2. If H contains 1 edge, G contains $K_{3,n}$ as a subgraph, but the two vertices of X sharing an edge have degree 4.
3. If H contains 2 edges, H is a path of length 2, so G is isomorphic to the fan graph $F_{3,n}$.
4. If H contains 3 edges, G is isomorphic to J_n.

The fourth scenario above, in which H is a cycle and G is isomorphic to a join, will now be verified.

Theorem 2.9. *For all $n \in \mathbb{N}$, J_n is VSC.*

Proof. Let J_n have any valid orientation and let $C = C_3$ denote the cycle of J_n. If C is a directed cycle, it is easy to see that J_n is strongly connected. Now assume that C is not a directed cycle and let $V(C) = \{x, y, z\}$. Since C is not directed, the subgraph of J_n induced by $V(C)$ contains both a source and a sink. Without loss of generality, x is a source and z is a sink. Since J_n is source-free and sink-free, there must be some vertex $a \in V(J_n)$ such that the edge ax is directed toward x, and a vertex $b \in V(J_n)$ such that the edge bz is directed towards b.

If $a = b$, then $\{x, y, z, a\}$ induce an isomorphic copy of J_1, say J, which contains no source or sink, and hence is strongly connected. Since every vertex of $J_n \setminus J$ must send one edge to J and receive one edge from J, it follows that J_n is strongly connected.

Now assume $a \neq b$. Note that there is a directed path from a to every vertex of C, and from each vertex of C to b, as indicated in Figure 5 applied to J_4. Similarly, for any $c \in J_n \setminus \{a, b\}$, there is a directed path from a to c (concatenating an edge directed from C to c with a path from a to the appropriate vertex on C) and a directed path from c to b (concatenating an appropriate directed path from C to b with an edge directed from c to C).

Next, we show that there exists a directed path from b to a. Note that one of ya and za must be directed toward a; it follows that if xb is directed toward x, then there is a directed path from b to a. So, we may assume xb is directed toward b. Since b is not a sink, the edge yb must be directed from b to y. Now, either ya is directed from y to a (in which case bya is a directed path from b to a) or za is directed from z to a (in which case $byza$ is a directed path from b to a).

Now, given any vertices c, d of J_n, concatenate a directed path from c to b with a directed path from b to a and a directed path from a to d. This produces a directed walk from c to d (which is either equal to or contains a directed path from c to d.) The same argument with the roles of c, d reversed establishes strong connectivity.

Thus, J_n is strongly connected for every valid orientation, and hence is VSC.

□

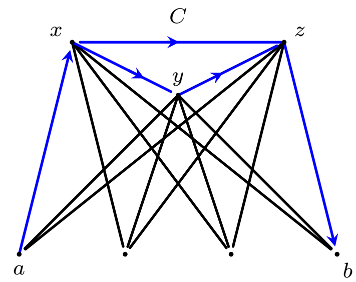

Figure 5: J_4 with $a \neq b$ and C not a directed cycle.

The graphs in the remaining three cases from Theorem 2.1 can be shown to be VSC using similar and less involved arguments and are left to the reader.

3 Classifying all anti-VSC graphs

The four examples of VSC families from Section 2 completely characterize all finite simple bridgeless connected graphs whose vertices have at least degree 3 and do not contain two disjoint cycles, as shown in [1].

Using this result, we will now show that all other bridgeless connected graphs whose vertices have at least degree 3 are anti-VSC. The proof will involve the use of ear decompositions, which are guaranteed to exist in all bridgeless connected graphs. See [3] for more information. The proof will also involve a process for isolating a subgraph in a manner that prevents the graph from being VSC.

Definition 3.1. Let H be a subgraph of a digraph G. If every edge of $G \setminus H$ that is incident to a vertex of H is oriented toward H - in effect, isolating H from the rest of G - we say that H is *quarantined*, and we say the edges of G that are directed toward the vertices of H are *quarantining edges*.

See Figure 6 for this argument applied to K_7. The quarantined subgraph is colored green; the quarantining edges are red; and all remaining edges are blue.

Theorem 3.2. *Let G be a finite bridgeless simple connected graph whose vertices have at least degree 3. Then G is anti-VSC if and only if G has two vertex-disjoint cycles.*

Proof. Suppose G is a finite bridgeless simple connected graph that does not have two vertex-disjoint cycles. Then by Theorem 2.1 and previous results, G is VSC. The forward direction follows via the contrapositive.

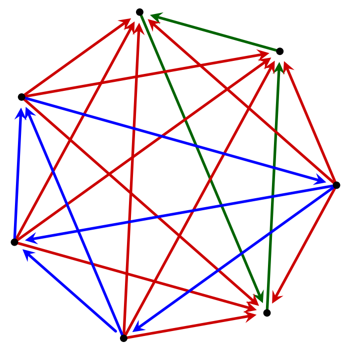

Figure 6: Quarantining a cycle in K_7.

Now assume that G has two vertex-disjoint cycles, C_α and C_ω. Direct the edges on both cycles. Now begin to create an ear decomposition of G from C_α. At each ear decomposition step, add a face of G to the component of G containing C_α that is both adjacent to this C_α component and disjoint from C_ω, and whose boundary is a directed path. Continue this process until this ear decomposition-generated subgraph of G containing C_α is maximal. Call this subgraph E_α. There are now two cases to consider:

1. All faces of $G \setminus \{E_\alpha \cup C_\omega\}$ are adjacent to both E_α and C_ω. In this case, all edges of $G \setminus \{E_\alpha \cup C_\omega\}$ have endpoints on both graph components; these undirected edges can be directed towards C_ω, and so C_ω is quarantined.

2. At least one of the faces of $G \setminus \{E_\alpha \cup C_\omega\}$ is adjacent to C_ω and not adjacent to E_α. In this case, begin to create an ear decomposition of G from C_ω that is disjoint from E_α. Continue until this subgraph of G, E_ω, is maximal. Then all edges of $G \setminus \{E_\alpha \cup E_\omega\}$ have endpoints on both graph components, and these undirected edges can again be directed towards C_ω, quarantining C_ω.

Note that in both cases, the specified edge orientations prevent the creation of sources or sinks. This verifies the reverse direction and the result then follows. □

See Figures 7 and 8 for examples of the arguments described in the proof of Theorem 3.2 applied to the Frucht graph.

We encourage interested readers to explicitly verify that any of the many named graph families not discussed here having two vertex disjoint cycles are in fact anti-VSC. Similarly, curious readers can explicitly show that named families of graphs not having disjoint vertex cycles are VSC.

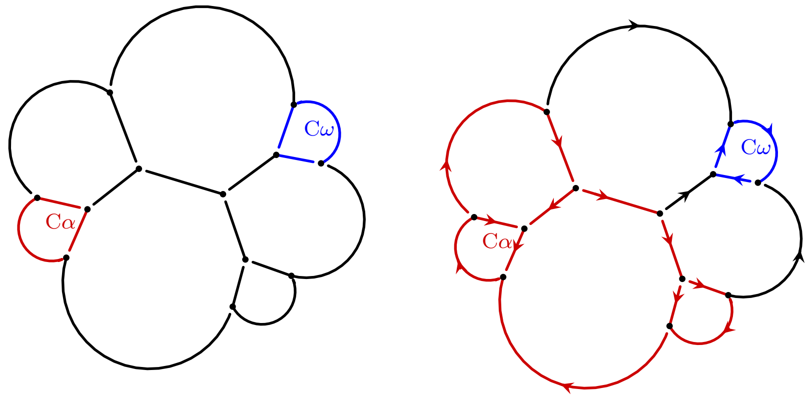

Figure 7: Disjoint cycles C_α, C_ω in the Frucht graph and the maximal subgraph E_α.

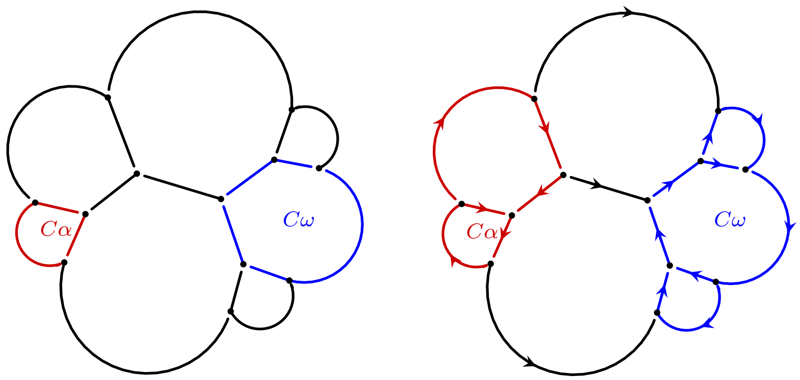

Figure 8: Disjoint cycles C_α, C_ω in the Frucht graph and the maximal subgraphs E_α and E_ω.

Acknowledgments

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