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Vacillating Parking Functions and the Fibonacci Numbers

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Abstract

Vacillating parking functions are parking functions in which a car only tolerates parking in its preferred spot, in the spot behind its preferred spot, or in the spot ahead of its preferred spot, which they check precisely in that order. Our main result characterizes the possible permutations that arise as parking outcomes from the parking process of nondecreasing vacillating parking functions, which are vacillating parking functions in which every car prefers a spot at least the preference of the previous car. We show that a permutation is the outcome of a nondecreasing vacillating parking function if and only if the permutation is a product of commuting adjacent transpositions. This readily implies that the number of distinct permutations arising as outcomes of nondecreasing vacillating parking functions is a Fibonacci number. We also show that the number of nondecreasing vacillating parking functions that have a fixed outcome consisting of k commuting adjacent transpositions is always a power of two. We conclude by using these results to give a new formula for the number of nondecreasing vacillating parking functions.

1 Introduction

Imagine having a one-way street with exactly n parking spots numbered increasingly from 1 to n. At the beginning of the street, there is a queue of n cars, each with a preferred parking spot. For each i, we let $a_i \in [n] = \{1, 2, \ldots, n\}$ denote the preference of car i. Cars enter the street in order $i = 1, 2, \ldots, n$ and attempt to park. For each $i \in [n]$, car i attempts to park in spots $a_i, a_i - 1, a_i + 1$, in this order and if those spots exist. In this way, for each $i \in [n]$, car i "vacillates" in its attempt to park, by first checking its preferred spot a_i, then by checking the spot immediately behind its preferred spot $a_i - 1$ (if it exists), and lastly by checking the spot immediately ahead of its preferred spot $a_i + 1$ (if it exists). If none of those

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spots are available to park in, then we say parking fails. If all cars are able to park based on this vacillating parking rule, then we say that the preference list $\alpha = (a_1, a_2, \ldots, a_n)$ is a *vacillating parking function*. A vacillating parking function $\alpha = (a_1, a_2, \ldots, a_n)$ is called a *nondecreasing vacillating parking function* if $a_i \leq a_{i+1}$ for all $1 \leq i \leq n-1$. We let VPF_n be the set of vacillating parking functions and we let $\mathrm{VPF}_n^{\uparrow}$ denote the set of nondecreasing vacillating parking functions.

Not every preference list is a vacillating parking function. For example, $(4,3,3,1,4)$ is a vacillating parking function and we illustrate the final parking outcome (the order that the cars ultimately park on the street) in Figure 1. On the contrary, one can readily check that the preference list $(5,4,3,3,1,1)$ is not a vacillating parking function since the last car fails to park.

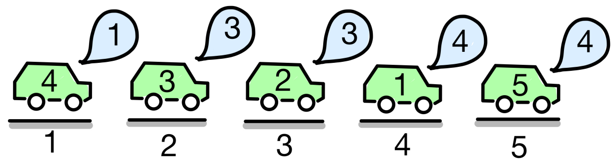

Figure 1: The parking outcome of the vacillating parking function $(4,3,3,1,4)$.

Fang, Harris, Kamau, and Wang were the first to define and study vacillating parking functions [2]. Among their contributions, Fang et al. give a way to determine if a preference list is a vacillating parking function and a recursive formula for the number of vacillating parking functions. They also show [2, Corollary 3.4] that the number of nondecreasing vacillating parking functions of length n satisfies

$$|\mathrm{VPF}_n^{\uparrow}| = \frac{(1+\sqrt{2})^n + (1-\sqrt{2})^n}{2}. \tag{1.1}$$

As n grows, the sequence $|\mathrm{VPF}_n^{\uparrow}|$ is precisely the numerators of continued fraction convergents to $\sqrt{2}$, which are known as the Pell-Lucas numbers, see sequence A001333 in the OEIS [5].

Our main result is concerned with the possible *outcomes* of nondecreasing vacillating parking functions. Recall that if all cars park under a given parking preference α, then the resulting parking configuration is called the *parking outcome* of α, which is a permutation of the numbers in the set $[n]$. We let $\mathfrak{S}_n$ denote the set of permutations of $[n]$ and write permutations in one-line notation $\pi = \pi(1)\pi(2)\cdots\pi(n)$. For any $1 \leq i \leq n-1$, we also make use of the notation s_i to denote the adjacent transposition that swaps i and $i+1$. We now introduce the outcome function $\mathcal{O} : \mathrm{VPF}_n \to \mathfrak{S}_n$ defined by $\mathcal{O}(\alpha) = \pi(1)\pi(2)\cdots\pi(n)$ where $\pi(i) = j$ means that given the preference list α, car j parked in spot i on the street. In Figure 1, the outcome of the vacillating parking function $\alpha = (4,3,3,1,4)$ meant spots 1, 2, 3, 4, 5 were occupied by the cars 4, 3, 2, 1, 5, in that order. Hence, $\mathcal{O}((4,3,3,1,4)) = 43215$.

Our main result characterizes the possible parking outcomes arising from the parking process of nondecreasing vacillating parking functions.

Theorem 1.1. *Let* $\pi \in \mathfrak{S}_n$*. Then* $\pi = \mathcal{O}(\alpha)$ *for some* $\alpha \in \mathrm{VPF}_n^{\uparrow}$ *if and only if* $\pi = s_{i_1}s_{i_2}\cdots s_{i_k}$ *is a product of commuting adjacent transpositions. That is, for all* $n \geq 1$*,*

$$\{\mathcal{O}(\alpha) : \alpha \in \mathrm{VPF}_n^{\uparrow}\} = \{\pi \in \mathfrak{S}_n : \pi \textit{ is a product of commuting adjacent transpositions}\}\,.$$

Theorem 1.1 readily implies that the number of distinct outcomes of nondecreasing vacillating parking functions in $\mathrm{VPF}_n^{\uparrow}$ is given by F_n, the nth Fibonacci number, where $F_0 = 1$, $F_1 = 2$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. We state this result formally.

Corollary 1.2. *For all* $n \geq 1$*,* $|\{\mathcal{O}(\alpha) : \alpha \in \mathrm{VPF}_n^{\uparrow}\}| = F_n$*, the* n*th Fibonacci number.*

Next we fix π to be a commuting product of adjacent transpositions and give a count for the number of nondecreasing vacillating parking functions whose parking outcome is π.

Theorem 1.3. *Fix* $\pi \in \mathfrak{S}_n$*, a commuting product of* k *adjacent transpositions. If* $\mathcal{O}^{-1}(\pi) = \{\alpha \in \mathrm{VPF}_n^{\uparrow} : \mathcal{O}(\alpha) = \pi\}$*, then*

$$|\mathcal{O}^{-1}(\pi)| = \begin{cases} 2^{n-2k} & \textit{if } \pi(1) = 2 \textit{ and } \pi(2) = 1 \\ 2^{n-2k-1} & \textit{otherwise.} \end{cases}$$

With the above results at hand, we conclude by giving a new formula for the number of nondecreasing vacillating parking functions.

Theorem 1.4. *If* $n \geq 1$*, then*

$$|\mathrm{VPF}_n^{\uparrow}| = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-k}{k-1} 2^{n-2k} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} 2^{n-1-2k}.$$

It is worth noting that while Theorem 1.4 and Equation (1.1) both provide formulas for counting nondecreasing vacillating parking functions of length n, it is clear in the formulation of Theorem 1.4 that this is an integer sequence, but this property is not so obvious from Equation (1.1).

Remark 1.5. Vacillating parking functions are inspired by parking functions, which are those preference lists allowing all cars to park when they park in their preference if available, or they park in the first spot available past their preference. Parking functions were introduced by Konheim and Weiss [3] who showed there are $(n+1)^{n-1}$ parking functions with n cars and n parking spots. Since then, there have been numerous variants of parking functions and parking functions have been shown to have connections to a wide variety of mathematical areas and to interesting integer sequences. For example, when keeping track of the number of lucky cars, those cars that park in their preferences, Aguillon et al. showed a connection to the tower of Hanoi [1], while Harris et al. showed a connection to the `Quicksort` algorithm [4]. It is remarkable how simple parking problems can lead to such nice mathematics.

2 Proofs

Not every permutation can be the outcome of a nondecreasing vacillating parking function and we illustrate this in the next example.

Example 2.1. *Can the permutation* 231 *be the outcome of a nondecreasing parking function?* Suppose this is the case and the preference list is given by (a_1, a_2, a_3) with $a_1 \leq a_2 \leq a_3$. The outcome being 231 implies that car 1 parked in spot 3, car 2 in spot 1, and car 3 in spot 2. In order for car 1 to park in spot 3 it must have preferred that parking spot. Hence$a_1 = 3$. Car 2 parked in spot 1, and at this point in the parking process, spot 2 would have been empty. So, the only way for car 2 to park in spot 1 is if it prefers that parking spot. Hence, $a_2 = 1$. But now $a_1 = 3$ and $a_2 = 1$, and so $a_1 \not\leq a_2$. Thus 231 is not the outcome of any nondecreasing vacillating parking function.

We are now ready to prove our first result.

Proof of Theorem 1.1. To begin, let us assume that $\pi = \mathcal{O}(\alpha)$ for some $\alpha \in \mathrm{VPF}_n^{\uparrow}$. In the forward direction of the proof we proceed by contradiction. Suppose that π has cycle notation $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$, where we omit all fixed points, and which does not consist of commuting adjacent transpositions. Then consider x, y to be the smallest distinct indices $1 \leq x, y \leq n-1$ such that $s_x s_y \neq s_y s_x$. Without loss of generality assume that $x < y$. Then $s_x s_y$ must contain consecutive integers and $y = x+1$ so that $s_x s_y = s_x s_{x+1} = (x, x+1)(x+1, x+2) = (x, x+1, x+2)$ is a length three cycle and this means that $\pi(x) = x+1$, $\pi(x+1) = x+2$, and $\pi(x+2) = x$. But recall that $\pi(a) = b$ means that car a parked in spot b. Hence

- $\pi(x) = x+1$, implies that car x with preference a_x parked in spot $x+1$,
- $\pi(x+1) = x+2$, implies that car $x+1$ with preference a_{x+1} parked in spot $x+2$, and
- $\pi(x+2) = x$, implies that car $x+2$ with preference a_{x+2} parked in spot x.

However $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{VPF}_n^{\uparrow}$ and so $a_x \leq a_{x+1} \leq a_{x+2}$. We now recall [2, Lemma 3.2]: If $\alpha = (a_1, a_2, \ldots, a_n) \in \mathrm{VPF}_n^{\uparrow}$, then $i-1 \leq a_i \leq i+1$ for all $i \in [n]$. By this lemma we must have that

$$x - 1 \leq a_x \leq x + 1$$
$$x \leq a_{x+1} \leq x + 2 \quad (2.1)$$
$$x + 1 \leq a_{x+2} \leq x + 3 \quad (2.2)$$

By eq. (2.2), the only way for car $x+2$ to park in spot x is for it to have preferred spot $x+1$ (the smallest preference it could have) and back into spot x. Then by eq. (2.1), car $x+1$ would have to have preference $x \leq a_{x+1} \leq x+1$ (since α is nondecreasing). But if that is the case, then car $x+1$ would have preference $a_{x+1} = x$ or $a_{x+1} = x+1$. If $a_{x+1} = x$, then having arrived prior to car $x+2$, car $x+1$ would have parked in spot x, contradicting that $\pi = \mathcal{O}(\alpha)$. If $a_{x+1} = x+1$, then car $x+1$ would find spot $x+1$ occupied by car x, but would back into spot x. Once again contradicting that $\pi = \mathcal{O}(\alpha)$. Thus, no such index x exists and π is the product of commuting adjacent transpositions, as claimed.

For this converse direction we fix a permutation $\pi = s_{i_1}s_{i_2}\cdots s_{i_k}$, which is the product of k commuting adjacent transpositions with $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$ being nonconsecutive integers. We now construct a nondecreasing vacillating parking function which has parking outcome π. Consider $\alpha = (a_1, a_2, \ldots, a_n)$ where

$$a_j = \begin{cases} i_\ell + 1 & \text{if } j = i_\ell \text{ for some } 1 \leq \ell \leq k \\ j & \text{otherwise.} \end{cases}$$

Since α satisfies the inequality condition of [2, Lemma 3.2] we know that $\alpha \in \mathrm{VPF}_n^{\uparrow}$. The parking outcomes are as follows:

- car j parks in spot j whenever $j \neq i_\ell, i_\ell + 1$ for all $1 \leq \ell \leq k$,
- car i_ℓ parks in spot $i_\ell + 1$ for all $1 \leq \ell \leq k$, and
- car $i_\ell + 1$ parks in spot i_ℓ for all $1 \leq \ell \leq k$ (here the car backs up into that spot).

This shows that $\mathcal{O}(\alpha) = s_{i_1}s_{i_2}\cdots s_{i_k} = \pi$, as desired. □

Theorem 1.1 shows that the only possible outcomes of nondecreasing vacillating parking functions are permutations made up of commuting adjacent transpositions i.e., permutations of the form $s_{i_1}s_{i_2}\cdots s_{i_k}$, where $1 \leq i_1 < i_2 < \cdots < i_k \leq n-1$ are nonconsecutive integers. To count all such outcome permutations we need to count the number of ways to select nonconsecutive indices from the set $[n-1]$. The number of ways to select nonconsecutive integers from the set $[n-1]$ is F_n, the nth Fibonacci number. Here $F_0 = 1$ and $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for all $n \geq 2$. This proves Corollary 1.2.

Now that we know how many distinct outcomes there are, for each such outcome π we want to count how many nondecreasing vacillating parking functions yield that parking outcome. Before proceeding to prove the associated result (Theorem 1.3) we illustrate it via an example.

Example 2.2. Let $n = 4$. By Corollary 1.2 there are $F_4 = 5$ outcomes of nondecreasing vacillating functions, which one can confirm they are $e = 1234, s_3 = 1243, s_2 = 1324, s_1 = 2134, s_1s_3 = 2143$. For each of these outcomes we now count the number of preferences each car could have so as to end up parking in those spots using a nondecreasing vacillating parking function.

- For $e = 1234$, car 1 has only option for spot 1, while each car after that can either prefer the spot they parked in or the previous. This gives a total of $1 \cdot 2 \cdot 2 \cdot 2 = 8$ possible nondecreasing vacillating parking functions with outcome $e = 1234$. Note $e = 1234$ is the product of $k = 0$ adjacent transpositions and $e(1) = 1 \neq 2$, so the count agrees with $2^{n-2k-1} = 2^{4-1} = 8$, as given in Theorem 1.3.

- For $s_3 = 1243$, car 1 has only option for spot 1, car 2 can either prefer the spot it parked in or the previous, and cars 3 and 4 can only prefer spot 4. This gives a total of $1 \cdot 2 \cdot 1 \cdot 1 \cdot = 2$ possible nondecreasing vacillating parking functions with outcome s_3. Note $s_3 = 1243$ is the product of $k = 1$ adjacent transposition and $s_3(1) = 1 \neq 2$, so the count agrees with $2^{n-2k-1} = 2^{4-2-1} = 2$, as given in Theorem 1.3.

- For $s_2 = 1324$, car 1 has only option for spot 1, cars 2 can only prefer spot 3, car 3 cannot prefer spots 1 or 2, as otherwise this would not be a nondecreasing vacillating parking function, so car 3 can only prefer spot 3, and car 4 can prefer either spot 3 or 4. This gives a total of $1 \cdot 1 \cdot 1 \cdot 2 = 2$ possible nondecreasing vacillating parking functions with outcome s_2. Note $s_2 = 1324$ is the product of $k = 1$ adjacent transposition and $s_2(1) = 1 \neq 2$, so the count agrees with $2^{n-2k-1} = 2^{4-2-1} = 2$, as given in Theorem 1.3.

- For $s_1 = 2134$, cars 1 and 2 can only prefer spot 2, so as to come from a nondecreasing vacillating parking function, while cars 3 and 4 prefer either the spot they park in or the previous. This gives a total of $1 \cdot 1 \cdot 2 \cdot 2 = 4$ possible nondecreasing vacillating parking functions with outcome s_1. Note $s_1 = 2134$ is the product of $k = 1$ adjacent transposition and $s_2(1) = 2$ and $s_2(2) = 1$, so the count agrees with $2^{n-2k} = 2^{4-2} = 4$, as given in Theorem 1.3.

- For $s_1 s_3 = 2143$, cars 1 and 2 can only prefer spot 2, and cars 3 and 4 can only prefer spot 4. This gives a total of $1 \cdot 1 \cdot 1 \cdot 1 = 1$ possible nondecreasing vacillating parking function with outcome $s_1 s_3$. Note $s_1 s_3 = 2143$ is the product of $k = 2$ adjacent transposition and $s_2(1) = 2$ and $s_2(2) = 1$, so the count agrees with $2^{n-2k} = 2^{4-2(2)} = 1$, as given in Theorem 1.3.

The total count of these four cases is 17, which matches the value given by (1.1) setting $n = 4$.

In Example 2.2, we can observe that whenever $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$ is a product of k commuting adjacent transpositions, if $1 \in \{i_1, i_2, \ldots, i_k\}$, then there are 2^{n-2k} many nondecreasing vacillating parking functions with outcome π, and if $1 \notin \{i_1, i_2, \ldots, i_k\}$, then there are 2^{n-1-2k} many nondecreasing vacillating parking functions with outcome π. We now prove this result.

Proof of Theorem 1.3. Fix $\pi = s_{i_1} s_{i_2} \cdots s_{i_k} \in \mathfrak{S}_n$ to be a commuting product of k adjacent transpositions. Hence $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq [n-1]$ is a set of k nonconsecutive integers. We make the following observations:

1. Car 1 has a single parking preference, and it always parks where it prefers.

2. For any $j \in \{i_\ell, i_\ell + 1 : 1 \leq \ell \leq k\}$, $\pi(j) = j + 1$ and $\pi(j + 1) = j$. So car j parks in spot $j + 1$, and spot j would have been unoccupied upon its attempt to park (as spot j is where car $j + 1$ parks). Hence, car j could have only parked in spot $j + 1$ if it preferred that spot. Thus, car j has a unique preference. Moreover, car $j+1$ must also prefer spot $j+1$, as preferring spot j would imply that the vacillating parking function is not nondecreasing. Thus, car j and car $j + 1$ have a unique preference, whenever $j \in \{i_\ell, i_\ell + 1 : 1 \leq \ell \leq k\}$.

3. For any $j \notin \{1, i_\ell, i_\ell + 1 : 1 \leq \ell \leq k\}$, $\pi(j) = j$. So car j parks in spot j. As the vacillating parking function is nondecreasing, either car j prefers spot j or spot $j - 1$. Thus, car j has 2 preferences, whenever $j \notin \{1, i_\ell, i_\ell + 1 : 1 \leq \ell \leq k\}$.

Taking the product over these potential preferences still depends on whether $1 \in I$ or not.

- When $1 \in I$, $i_1 = 1$, car 1 prefers and parks in spot 2. Then car 2 can only prefer spot 2, backing into spot 1, which ensures the vacillating parking function is nondecreasing. In this case, car 1 has a single preference, car 2 has a single preference, the cars $j > 2$ in $\{i_\ell, i_\ell + 1 : 2 \leq \ell \leq k\}$ each have a single preference, and the cars $j > 2$ in $\{i_\ell, i_\ell + 1 : 2 \leq \ell \leq k\}$ each have 2 preferences. Thus the count is given by

$$1 \cdot 1 \cdot 1^{2(k-1)} \cdot 2^{n-2-2(k-1)} = 2^{n-2k},$$

as claimed.

- If $1 \notin I$, then case 1 and 2 are disjoint and $i_1 \geq 2$. In this case, car 1 has a single preference, the cars $j > 1$ in $\{i_\ell, i_\ell + 1 : 1 \leq \ell \leq k\}$ each have a single preference, and the cars $j > 1$ in $\{i_\ell, i_\ell + 1 : 1 \leq \ell \leq k\}$ each have 2 preferences. Thus the count is given by

$$1 \cdot 1^{2k} \cdot 2^{n-1-2k} = 2^{n-2k-1},$$

as claimed. □

Using our previous results we now prove the formula in Theorem 1.4 for the number of nondecreasing vacillating parking functions.

Proof of Theorem 1.4. To begin we let $I = \{i_1 < i_2 < \cdots < i_k\} \subseteq [n-1]$ consist of k nonconsecutive integers and let $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$. Then for each such subset I we can construct an outcome $\pi = s_{i_1} s_{i_2} \cdots s_{i_k}$. The number of nondecreasing vacillating parking functions with this outcome depends on whether or not $1 \in I$ as we established in Lemma 1.3. Hence, we have that

$$|\mathrm{VPF}_n^{\uparrow}| = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left(\sum_{I \subseteq [n-1],\ 1 \in I,\ |I|=k} 2^{n-2k} \right) + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \left(\sum_{I \subseteq [n-1],\ 1 \notin I,\ |I|=k} 2^{n-1-2k} \right). \tag{2.3}$$

Note that the limits on the sums account for the minimal and maximal sizes of the sets I involved, which consist of nonconsecutive integers from the set $[n-1]$. We recall that the minimal and maximal number of nonconsecutive integers that can be selected from the set $[y]$ is 0 and $\lfloor \frac{y+1}{2} \rfloor$, respectively. Using this fact, we can simplify (2.3) as follows

$$|\mathrm{VPF}_n^{\uparrow}| = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} 2^{n-2k} \left(\sum_{I \subseteq [n-1],\ 1 \in I,\ |I|=k} 1 \right) + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} 2^{n-1-2k} \left(\sum_{I \subseteq [n-1],\ 1 \notin I,\ |I|=k} 1 \right) \tag{2.4}$$

$$= \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n-1-k}{k-1} 2^{n-2k} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} 2^{n-1-2k} \tag{2.5}$$

where (2.4) holds as the power of two is independent of the subset indexing the sum, and (2.5) follows from how many subsets satisfy the needed conditions. Namely, we recall that the number of ways to select x nonconsecutive integers from the set $[y]$ is given by $\binom{x-y+1}{x}$. This concludes the proof. □

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