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Generalization of a formula for marked plane trees

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Abstract

Deutsch, Munarin and Rinaldi derived a formula for counting marked plane trees while investigating the enumeration of skew Dyck paths. The formula involves the Catalan numbers, which count plane trees among other classical combinatorial structures. In this paper, we generalize their result by enumerating families of noncrossing trees and recently introduced d-dimensional plane trees in which certain edges are marked. The resulting formulas are shown to also count: noncrossing trees that allow multi-edges, plane trees in which each internal vertex has outdegree at most 3 and some edges may be marked, and ternary trees in which certain edge type are colored using two colors. These generalizations provide new combinatorial interpretations and extend the scope of the original enumeration.

1 Introduction

Consider plane trees in which the vertices, labeled in counterclockwise order, are placed on the circumference of a circle and the edges are drawn inside the circle without crossings. These were introduced by Noy in [4] and are commonly referred to as *noncrossing trees*. A plane tree representation of noncrossing trees, known as the (l, r)-*representation*, was introduced by Panholzer and Prodinger in 2002 [6]. In this representation, vertex 1 is the root of the plane tree. If the label of a child vertex is greater than that of its parent, it receives label r; otherwise, it receives label l. Figure 1 illustrates a noncrossing tree with 8 vertices alongside its corresponding (l, r)-representation. In any plane tree (and by extension, noncrossing trees), a vertex i that appears on a lower level than its adjacent vertex j is considered a child of j. The number of children of a vertex is called its *degree*, and vertices of degree zero are called *leaves*. The longest path from the root to a leaf is

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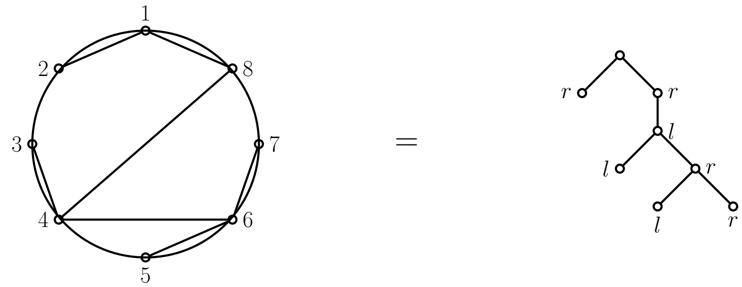

Figure 1: A noncrossing tree on 8 vertices alongside its (l, r)-representation.

the *height* of the tree. A *forest* is a collection of disjoint trees. Noncrossing trees have been studied and enumerated with respect to root degree [2], number of leaves, forests, degree sequences [3], and other parameters.

A significant technique in the enumeration of noncrossing trees is the butterfly decomposition, introduced by Flajolet and Noy [3]. A *butterfly* is defined as an ordered pair of noncrossing trees (the 'wings'), where each wing is itself a noncrossing tree. In the (l, r)-representation, vertices labeled l belong to the left wing and those labeled r to the right wing. This concept was generalized by Okoth and Kasyoki [5] to define *d-dimensional plane trees*. In this generalization, a butterfly consists of d ordered wings, each being a plane tree. The results in [5] unify the theories of plane and noncrossing trees: plane trees are 1-dimensional, and noncrossing trees are 2-dimensional plane trees. Figure 2 displays a 3-dimensional plane tree on 19 vertices. Here, the labels 1, 2, and 3 correspond to the wings of each butterfly, ordered from left to right.

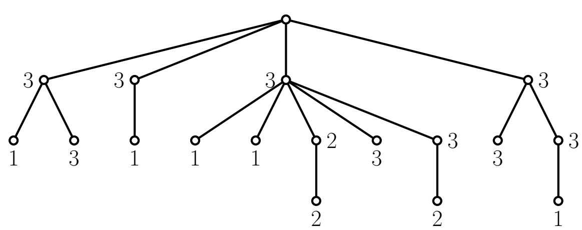

Figure 2: A 3-dimensional plane tree on 19 vertices.

In [1], Deutsch and his co-authors introduced plane trees in which a rightmost edge can be marked if it does not lead to a leaf. They called these plane trees as *marked plane trees*. They established that the number of marked plane trees with n vertices is given by the formula

$$\sum_{k=1}^{n-1} \binom{n-2}{k-1} c_k, \tag{1.1}$$

where

$$c_k = \frac{1}{k+1}\binom{2k}{k}$$

is the k^{th} Catalan number, which enumerates plane trees with k vertices, among many other classical combinatorial structures listed in sequence A000108 of the Online Encyclopaedia of Integer Sequences (OEIS) [8]. The authors further showed that equation (1.1) also counts skew Dyck paths, hex trees, and 3-Motzkin paths. In a subsequent collection of articles [7], Prodinger extended these insights by presenting a variety of other combinatorial structures that are enumerated by (1.1). These are recorded in entry A002212 of the OEIS [8]. We acknowledge that Prodinger's work was not only foundational but also elegantly accessible and it directly inspired the current investigation. A natural question that arises from this line of inquiry is: What combinatorial structures are counted by equation (1.1) if c_k is replaced with

$$c_k = \frac{1}{2k+1}\binom{3k}{k},$$

the enumeration formula for noncrossing trees on $k+1$ vertices? More generally, one may ask: Which structures are counted by (1.1) if

$$c_k = \frac{1}{dk+1}\binom{(d+1)k}{k}$$

for a fixed integer $d \geq 1$? In this article, we tackle these questions by identifying and enumerating new combinatorial families whose counting sequences arise from this generalized sum. We make use of the following theorem in our enumeration.

Theorem 1.1 ([9, Lagrange Inversion Formula]). *Let $T(x)$ be a generating function satisfying the functional equation $T(x) = z\phi(T(x))$, where $\phi(0) \neq 0$. Then, $n[x^n]T^k = k[t^{n-k}]\phi(t)^n$.*

This paper is organized as follows. In Sections 2 and 3, we introduce families of noncrossing trees and d-dimensional plane trees with some marked edges that are enumerated by the formulas

$$\sum_{k=1}^{n-1}\binom{n-2}{k-1}\cdot\frac{1}{2k+1}\binom{3k}{k} \tag{1.2}$$

and

$$\sum_{k=1}^{n-1}\binom{n-2}{k-1}\cdot\frac{1}{dk+1}\binom{(d+1)k}{k},$$

respectively. We further enumerate these classes of trees with respect to the number of vertices, marked edges and leaves. In Section 4, we introduce three distinct combinatorial structures; noncrossing trees with multiple edges, plane trees with some labeled edges, and ternary trees in which certain edges appear in two distinct colors, and show that all are enumerated by the expression in (1.2). Finally, Section 5 gives a summary of the results and poses open enumerative questions for future investigation.

2 Enumeration of a kind of noncrossing Trees

2.1 Number of vertices

Consider a noncrossing tree in which in its (l, r)-representation, an edge in a rightmost subtree can be marked if it does not lead to a leaf and if it leads to a vertex in which all vertices (root not included) of the subtree rooted at the vertex are either all labeled r or all labeled l. We shall call these trees as *partially marked noncrossing trees.* See Figure 3 for an example of partially marked noncrossing tree.

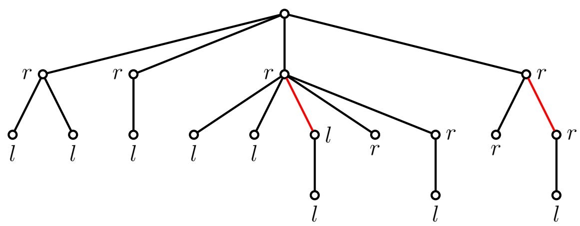

Figure 3: A partially marked noncrossing tree on 19 vertices.

Let $\mathcal{M}$ be a family of partially marked noncrossing trees. Symbolically, this is expressed as shown in Figure 4, where $\mathcal{M}^2$ represents a butterfly of noncrossing trees.

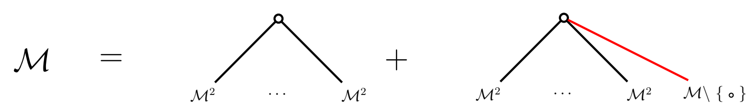

Figure 4: Symbolic representation of partially marked noncrossing trees.

Let $M(z) = M$ be the generating function for partially marked noncrossing trees with z marking the number of vertices. By the symbolic representation, the generating function satisfies the functional equation

$$M(z) = \frac{z}{1 - \frac{M(z)^2}{z}} + \frac{z(M(z) - z)}{1 - \frac{M(z)^2}{z}}.$$

This equation can be rewritten as

$$zM = M^3 + z^2M + z^2 - z^3. \tag{2.1}$$

Now, let $M = z + zv$. Then (2.1) becomes

$$v = z\left((1 + v)^3 + v\right). \tag{2.2}$$

We extract $[z^n]\,M$. We use Lagrange Inversion Formula [9] as follows:

$$\begin{aligned}
[z^n]\,M = \left[z^{n-1}\right] v &= \frac{1}{n-1}\left[v^{n-2}\right]\left((1+v)^3+v\right)^{n-1}\\
&= \frac{1}{n-1}\left[v^{n-2}\right]\sum_{k\geq 0}\binom{n-1}{k}(1+v)^{3k}v^{n-1-k}\\
&= \frac{1}{n-1}\left[v^{n-2}\right]\sum_{k,j\geq 0}\binom{n-1}{k}\binom{3k}{j}v^{n-k+j-1}\\
&= \frac{1}{n-1}\sum_{k\geq 0}\binom{n-1}{k}\binom{3k}{k-1}\\
&= \sum_{k=1}^{n-1}\binom{n-2}{k-1}\frac{1}{2k+1}\binom{3k}{k}.
\end{aligned}$$

The following theorem gives the result we have just obtained.

Theorem 2.1. *The number of partially marked noncrossing trees on n vertices is given by*

$$\sum_{k=1}^{n-1}\binom{n-2}{k-1}\frac{1}{2k+1}\binom{3k}{k}. \tag{2.3}$$

One can also ask what the expression,

$$\binom{n-2}{k-1}\frac{1}{2k+1}\binom{3k}{k}, \tag{2.4}$$

could be counting in terms of partially marked noncrossing trees. We provide an answer to this question in the next subsection.

2.2 Marked edges

In the sequel, we obtain the number of partially marked noncrossing trees with a given number of vertices and marked edges.

Theorem 2.2. *The number of partially marked noncrossing trees on n vertices and m marked edges is given by*

$$\frac{1}{n-1}\binom{n-1}{m}\binom{3n-3m-3}{n-m-2}. \tag{2.5}$$

Proof. Let $M(z,e) = M$ be the bivariate generating function for partially marked noncrossing trees with z marking the number of vertices and e marking marked edges. We have

$$M(z) = \frac{z}{1-\frac{M(z)^2}{z}} + \frac{ez(M(z)-z)}{1-\frac{M(z)^2}{z}}.$$

This can be rewritten as

$$zM = M^3 + ez^2M + z^2 - ez^3. \tag{2.6}$$

Now, let $M = z + zv$ so that (2.6) becomes

$$v = z\left((1+v)^3 + ev\right). \tag{2.7}$$

We extract $[z^n e^m]\, M$ by means of Lagrange Inversion Formula [9] and equation (2.7):

$$\begin{aligned}
[z^n e^m]\, M = \left[z^{n-1} e^m\right] v &= \frac{1}{n-1}\left[v^{n-2} e^m\right]\left((1+v)^3 + ev\right)^{n-1} \\
&= \frac{1}{n-1}\left[v^{n-2} e^m\right]\sum_{i\geq 0}\binom{n-1}{i}(1+v)^{3(n-1-i)} e^i v^i \\
&= \frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2}\right](1+v)^{3(n-m-1)} \\
&= \frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2}\right]\sum_{j\geq 0}\binom{3(n-m-1)}{j} v^j \\
&= \frac{1}{n-1}\binom{n-1}{m}\binom{3(n-m-1)}{n-m-2}.
\end{aligned}$$

□

Setting $k = n - m - 1$ in (2.5), we see that (2.4) gives the number of partially marked noncrossing trees with n vertices and $n - k - 1$ marked edges or put simply as k unmarked edges.

Corollary 2.3. *There are*

$$\frac{1}{2n-1}\binom{3n-3}{n-1}$$

noncrossing trees with n vertices.

Proof. Set $m = 0$ in (2.5). □

2.3 Leaves

Our aim in this subsection is to find closed formula for the number of partially marked noncrossing trees with a given number of leaves.

Theorem 2.4. *There are*

$$\frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\sum_{j=0}^{\ell-1}\binom{n-m-1}{j}\binom{n-m-\ell-1}{\ell-j-1} 2^{n-m-2\ell+j} \tag{2.8}$$

partially marked noncrossing trees on n vertices with m marked edges and ℓ leaves.

Proof. Let $M(z,u,e)=M$ be the trivariate generating function for partially marked noncrossing trees with z,u and e marking vertices, leaves and marked edges respectively. The expression for a butterfly is thus $M^2/z+zu-z$ and the required equation for the generating function is given by

$$M=\frac{z}{1-\frac{M^2}{z}-zu+z}+\frac{ze(M-z)}{1-\frac{M^2}{z}-zu+z}.$$

This reduces to

$$zM=M^3+z^2uM-z^2M+z^2+z^2eM-ez^3.$$

Let $M=z+zv$. We have

$$v=z\left((1+v)^3+(u-1)(1+v)+ev\right)$$

or

$$v=z((1+v)(v(2+v)+u)+ev). \tag{2.9}$$

We extract $\left[z^n u^\ell e^m\right]M$ using equation (2.9):

$$\begin{aligned}
[z^n u^\ell e^m]M &= \left[z^{n-1}u^\ell e^m\right]v\\
&=\frac{1}{n-1}\left[v^{n-2}u^\ell e^m\right]((1+v)(v(2+v)+u)+ev)^{n-1}\\
&=\frac{1}{n-1}\left[v^{n-2}u^\ell e^m\right]\sum_{i\geq 0}\binom{n-1}{i}(1+v)^i(v(2+v)+u)^i v^{n-i-1}e^{n-i-1}\\
&=\frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2}u^\ell\right](1+v)^{n-m-1}(v(2+v)+u)^{n-m-1}\\
&=\frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2}u^\ell\right]\\
&\sum_{j,k\geq 0}\binom{n-m-1}{j}\binom{n-m-1}{k}(2+v)^{n-m-k-1}u^k v^{n-m-k+j-1}\\
&=\frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\left[v^{n-m-2}\right]\\
&\sum_{j\geq 0}\binom{n-m-1}{j}(2+v)^{n-m-\ell-1}v^{n-m-\ell+j-1}\\
&=\frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\left[v^{n-m-2}\right]\\
&\sum_{i,j\geq 0}\binom{n-m-1}{j}\binom{n-m-\ell-1}{i}2^{n-m-\ell-i-1}v^{n-m-\ell+i+j-1}.
\end{aligned}$$

So,

$$[z^n u^\ell e^m]M = \frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\sum_{j=0}^{\ell-1}\binom{n-m-1}{j}\binom{n-m-\ell-1}{\ell-j-1}2^{n-m-2\ell+j}.$$

□

The following result follows upon setting $e = 1$ in (2.9) and extracting the coefficient of $z^{n-1}u^\ell$ in v.

Corollary 2.5. *The number of marked noncrossing trees on n vertices with ℓ leaves is given by*

$$\frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\sum_{j=0}^{\ell-1}\binom{n-\ell-1}{i}\binom{n-i-1}{j}\binom{n-i-1}{\ell-j-1}2^{n-2\ell+j-i}$$

If we set $m = 0$ in (2.8), the result is

$$\frac{1}{n-1}\binom{n-1}{\ell}\sum_{j=0}^{\ell-1}\binom{n-1}{j}\binom{n-\ell-1}{\ell-j-1}2^{n-2\ell+j}$$

which counts noncrossing trees on n vertices with ℓ leaves. This formula was first obtained by Flajolet and Noy in [3].

3 Generalization to d-dimensional plane trees

3.1 Number of vertices

Consider a d-dimensional plane tree satisfying the following additional property: an edge in a rightmost subtree can be marked if it does not lead to a leaf and if it leads to a vertex in which all vertices (root not included) of the butterfly rooted at the vertex are all in the same wing, i.e., the marked edge leads to a vertex which is a root of a butterfly with just one wing. Let us coin the name, *partially marked d-dimensional plane trees* for these trees. Figure 5 is a depiction of a partially marked 3-dimensional plane tree.

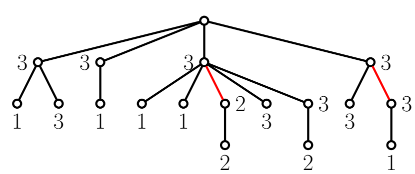

Figure 5: A partially marked 3-dimensional plane tree on 19 vertices.

We now obtain a functional equation satisfied by the generating function for these trees. Let $\mathcal{D}$ be a family of partially marked d-dimensional plane trees. The trees are symbolically expressed as shown in Figure 6 where $\mathcal{D}^d$ is a butterfly with d wings.

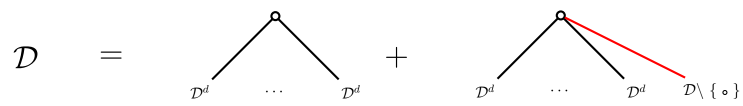

Figure 6: Symbolic representation of partially marked d-dimensional plane trees.

Let $D(z) = D$ be the generating function for partially marked d-dimensional plane trees where z marks a vertex. Using the symbolic representation, we find that the generating function $D(z)$ is expressed in terms of itself as

$$D(z) = \frac{z}{1 - \frac{D(z)^d}{z^{d-1}}} + \frac{z(D(z) - z)}{1 - \frac{D(z)^d}{z^{d-1}}}.$$

This equation reduces to

$$z^{d-1}D = D^{d+1} + z^d D + z^d - z^{d+1}. \tag{3.1}$$

Equation (3.1) is not in a form we can use Lagrange Inversion Formula [9], so we let $D = z + zv$. So, we have

$$v = z\left((1+v)^{d+1} + v\right). \tag{3.2}$$

Let us extract $[z^n]\,D$ by applying Lagrange Inversion Formula on (3.2):

$$\begin{aligned}
[z^n]\,D = \left[z^{n-1}\right]v &= \frac{1}{n-1}\left[v^{n-2}\right]\left((1+v)^{d+1} + v\right)^{n-1} \\
&= \frac{1}{n-1}\left[v^{n-2}\right]\sum_{k\geq 0}\binom{n-1}{k}(1+v)^{(d+1)k}v^{n-1-k} \\
&= \frac{1}{n-1}\left[v^{n-2}\right]\sum_{k,j\geq 0}\binom{n-1}{k}\binom{(d+1)k}{j}v^{n-k+j-1} \\
&= \frac{1}{n-1}\sum_{k\geq 0}\binom{n-1}{k}\binom{(d+1)k}{k-1} \\
&= \sum_{k=1}^{n-1}\binom{n-2}{k-1}\frac{1}{dk+1}\binom{(d+1)k}{k}.
\end{aligned}$$

Theorem 3.1 formally states this result.

Theorem 3.1. *There are*

$$\sum_{k=1}^{n-1}\binom{n-2}{k-1}\frac{1}{dk+1}\binom{(d+1)k}{k} \tag{3.3}$$

partially marked d-dimensional plane trees n vertices.

Note that marked plane trees and partially marked noncrossing trees are partially marked 1-dimensional and 2-dimensional plane trees respectively. The results of this section, therefore unify results of the previous section and also previous study on marked plane trees.

3.2 Marked edges

We find formula for partially marked d-dimensional plane trees with a given number of marked edges.

Theorem 3.2. *The number of partially marked d-dimensional plane trees with n vertices and m marked edges is*

$$\frac{1}{n-1}\binom{n-1}{m}\binom{(d+1)(n-m-1)}{n-m-2}. \tag{3.4}$$

Proof. Let $D(z, e) = D$ be the generating function for partially marked d-dimensional plane trees with z and e marking vertices and marked edges respectively. Based on the symbolic representation, the generating function is given by

$$D(z) = \frac{z}{1 - \frac{D(z)^d}{z^{d-1}}} + \frac{ez(D(z) - z)}{1 - \frac{D(z)^d}{z^{d-1}}}.$$

This is the same as

$$z^{d-1}D = D^{d+1} + ez^d D + z^d - ez^{d+1}. \tag{3.5}$$

Again as before, let $D = z + zv$ so that (3.5) reduces to

$$v = z\left((1+v)^{d+1} + ev\right). \tag{3.6}$$

By Lagrange Inversion Formula and equation (3.6), we get

$$\begin{aligned}
\left[z^n e^m\right] D = \left[z^{n-1}e^m\right] v &= \frac{1}{n-1}\left[v^{n-2}e^m\right]\left((1+v)^{d+1} + ev\right)^{n-1} \\
&= \frac{1}{n-1}\left[v^{n-2}e^m\right]\sum_{i\geq 0}\binom{n-1}{i}(1+v)^{(d+1)(n-1-i)}e^i v^i \\
&= \frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2}\right](1+v)^{(d+1)(n-m-1)} \\
&= \frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2}\right]\sum_{j\geq 0}\binom{(d+1)(n-m-1)}{j}v^j \\
&= \frac{1}{n-1}\binom{n-1}{m}\binom{(d+1)(n-m-1)}{n-m-2}.
\end{aligned}$$

□

If we let $k = n - m - 1$ in (3.4), we find a formula for the number of partially marked d dimensional plane trees with n vertices and k unmarked edges. The formula is given as

$$\binom{n-2}{k-1}\frac{1}{dk+1}\binom{(d+1)k}{k}.$$

If we set $m = 0$ in (3.4), we rediscover the following result that was first obtained by Okoth and Kasyoki in [5].

Corollary 3.3 ([5]). *There are*

$$\frac{1}{d(n-1)+1}\binom{(d+1)(n-1)}{n-1}$$

d-dimensional plane trees with n vertices.

We remark that on setting $d = 1$ and $d = 2$ in (3.4), we obtained the formulas for the number of marked plane trees and partially marked noncrossing trees on n vertices with m edges respectively.

3.3 Leaves

Theorem 3.4. *There are*

$$\frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\sum_{i=0}^{n-m-\ell-1}\binom{n-m-\ell-1}{i}\binom{(d+1)(n-m-1)-d\ell-di}{n-m-2}(-1)^i \tag{3.7}$$

partially marked d-dimensional plane trees on n vertices with m marked edges and ℓ leaves.

Proof. Let $D(z,u,e) = D$ be the trivariate generating function for partially marked d-dimensional plane trees where z, u and e are marking vertices, leaves and marked edges respectively. A butterfly is thus represented as $\frac{D^d}{z^{d-1}} + zu - z$. The generating function D is expressed as

$$D = \frac{z}{1-\frac{D^d}{z^{d-1}}-zu+z} + \frac{ze(D-z)}{1-\frac{D^d}{z^{d-1}}-zu+z}.$$

Simplification of the equation results in

$$z^{d-1}D = D^{d+1} + z^d uD - z^d D + z^d + z^d eD - ez^{d+1}.$$

As before let $D = z + zv$ so that

$$v = z\left((1+v)^{d+1} + (u-1)(1+v) + ev\right)$$

or

$$v = z\left((1+v)\left((1+v)^d - 1 + u\right) + ev\right). \tag{3.8}$$

We compute $\left[z^n u^\ell e^m\right] D$ using (3.8) and Lagrange Inversion Formula:

$$\left[z^n u^\ell e^m\right] D = \left[z^{n-1} u^\ell e^m\right] v$$
$$= \frac{1}{n-1}\left[v^{n-2} u^\ell e^m\right]\left((1+v)\left((1+v)^d - 1 + u\right) + ev\right)^{n-1}$$
$$= \frac{1}{n-1}\left[v^{n-2} u^\ell e^m\right]\sum_{i\geq 0}\binom{n-1}{i}(1+v)^i\left((1+v)^d - 1 + u\right)^i v^{n-i-1} e^{n-i-1}$$
$$= \frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2} u^\ell\right](1+v)^{n-m-1}\left((1+v)^d - 1 + u\right)^{n-m-1}$$
$$= \frac{1}{n-1}\binom{n-1}{m}\left[v^{n-m-2} u^\ell\right]\sum_{j,k\geq 0}\binom{n-m-1}{j}\binom{n-m-1}{k}\left((1+v)^d - 1\right)^{n-m-k-1} u^k v^j$$
$$= \frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\left[v^{n-m-2}\right]\sum_{j\geq 0}\binom{n-m-1}{j}\left((1+v)^d - 1\right)^{n-m-\ell-1} v^j$$
$$= \frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\left[v^{n-m-2}\right]$$
$$\sum_{i,j\geq 0}\binom{n-m-1}{j}\binom{n-m-\ell-1}{i}(-1)^i(1+v)^{d(n-m-\ell-i-1)} v^j$$
$$= \frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\left[v^{n-m-2}\right]$$
$$\sum_{i,j,k\geq 0}\binom{n-m-1}{j}\binom{n-m-\ell-1}{i}\binom{d(n-m-\ell-i-1)}{k}(-1)^i v^{j+k}$$
$$= \frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}$$
$$\sum_{i,k\geq 0}\binom{n-m-1}{n-m-k-2}\binom{n-m-\ell-1}{i}\binom{d(n-m-\ell-i-1)}{k}(-1)^i$$
$$= \frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\sum_{i\geq 0}\binom{n-m-\ell-1}{i}\binom{(d+1)(n-m-1) - d\ell - di}{n-m-2}(-1)^i.$$

□

On setting $e = 1$ in (3.8) and extracting the coefficient of $z^{n-1}u^\ell$ in v, we arrive at the following corollary.

Corollary 3.5. *The number of marked d-dimensional plane trees on n vertices with ℓ leaves is*

$$\frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\binom{n-\ell-1}{i}\binom{(d+1)(n-i-1) - d\ell}{n-2}(-1)^i.$$

Setting $m = 0$ in (3.7) results in

$$\frac{1}{n-1}\binom{n-1}{\ell}\sum_{i=0}^{n-\ell-1}\binom{n-\ell-1}{i}\binom{(d+1)(n-1)-d\ell-di}{n-2}(-1)^i$$

which enumerates d-dimensional plane trees on n vertices with ℓ leaves. Moreover, letting $d = 1$ in (3.7) and summing over all values of i, we get that there are

$$\frac{1}{n-1}\binom{n-1}{m}\binom{n-m-1}{\ell}\binom{n-m-1}{\ell-1} \tag{3.9}$$

marked plane trees with n vertices, m marked vertices and ℓ leaves. Further, summing over all values of ℓ in (3.9) results in the formula

$$\frac{1}{n-1}\binom{n-1}{m}\binom{2(n-m-1)}{n-m}$$

for the number of marked plane trees with n vertices and m marked edges. On setting $m = 0$ in (3.9), we get the Narayana number,

$$\frac{1}{n-1}\binom{n-1}{\ell}\binom{n-1}{\ell-1}$$

which gives the number of plane trees with n vertices and ℓ leaves.

4 Combinatorial structures related to partially marked noncrossing trees

4.1 Noncrossing multitrees

Let $N(z) = N$ be the generating function for noncrossing trees where z marks a vertex. The generating function for noncrossing trees with multiple edges is thus

$$N(z) = \frac{z}{1 - \frac{1}{1-z}\cdot\frac{N(z)^2}{z}}. \tag{4.1}$$

This equation can be simplified to give

$$zN(z) = N(z)^3 + z^2N(z) - z^3 + z^2. \tag{4.2}$$

Let $N = z(1 + v)$ so that equation (4.2) becomes

$$v = z\left((1+v)^3 + v\right)$$

which is the same as equation (2.2). Thus, the formula that counts noncrossing multitrees with n edges also counts partially marked noncrossing trees on n vertices. Figure 7 gives the 19 noncrossing multitrees with 3 edges displayed as plane trees using the (l, r) representation.

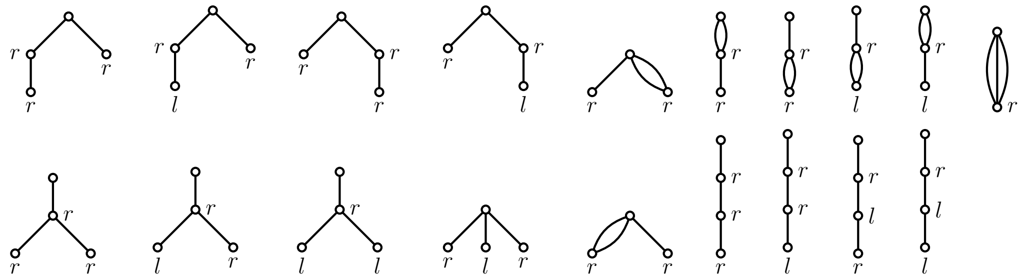

Figure 7: All the 19 noncrossing multitrees with 3 edges.

Many combinatorialists regard bijective proofs as particularly powerful, as they not only establish numerical equivalence between sets but also uncover deep structural relationships between combinatorial objects. Consequently, constructing a bijection between the set of noncrossing multitrees and the set of partially marked noncrossing trees becomes a compelling and necessary endeavor.

We can get a further refinement of equation (4.2). Let $N(z, e) = N$ be the generating function for noncrossing trees where z and e marks an edge and a multiple edge respectively. We have

$$N(z, e) = \frac{z}{1 - \frac{1}{1-ez} \cdot \frac{N^2}{z}}. \tag{4.3}$$

From equation (4.3), we get

$$zN = N^3 + ez^2N - ez^3 + z^2$$

which reduces to

$$v = z\left((1 + v)^3 + ev\right) \tag{4.4}$$

upon setting $N = z(1 + v)$. We also encountered equation (4.4) in the enumeration of partially marked noncrossing trees where e marks marked edges. So, the formula that counts noncrossing multitrees with n edges and m multiple edges also gives the number of partially marked noncrossing trees with n vertices and m marked edges.

Let $N(z, e, u)$ be the generating function for noncrossing multitrees with z, e and u marking edges, multiple edges and leaves respectively. Then,

$$N(z, e, u) = \frac{z}{1 - \frac{1}{1-ez} \cdot \left(\frac{N^2}{z} + zu - z\right)}. \tag{4.5}$$

Equation (4.5) can also be written as

$$zN = N^3 + (e + u - 1)z^2N - ez^3 + z^2$$

which becomes

$$v = z\left((1 + v)^3 + (u - 1)(1 + v) + ev\right) \tag{4.6}$$

on again setting $N = z(1+v)$. Equation (4.6) was also obtained in (2.9) as the generating function for partially marked noncrossing trees where v, e and u marks vertices, marked edges and leavves respectively. Thus, the formula that counts noncrossing multitrees with n edges, m multiple edges and ℓ leaves also counts partially marked noncrossing trees on n vertices with m marked edges and ℓ leaves.

4.2 Labeled Plane trees with outdegrees at most 3

Let $\mathcal{P}$ be a family of plane trees in which outdegree of each internal vertex is 1, 2 or 3, a right edge coming out of a vertex of degree 2 is labeled 1, 2 or 3 and an edge emanating from a vertex of outdegree 1 is labeled 1, 2, 3 or 4. In Figure 8, we get all the 19 labeled plane trees with 3 vertices and outdegrees of at most 3.

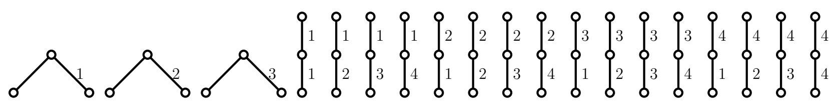

Figure 8: Labeled plane trees with 3 vertices and outdegrees at most 3.

The symbolic representation for these trees is given in Figure 9.

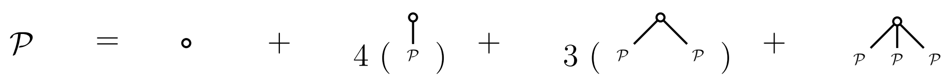

Figure 9: Symbolic representation of the plane trees.

Let $P(z) = P$ be the generating function for the trees in $\mathcal{P}$. So,

$$P(z) = z + 4zP(z) + 3zP(z)^2 + zP(z)^3.$$

The right hand side of this equation can be simplified to obtain

$$P = z((1+P)^3 + P). \tag{4.7}$$

Equation (4.7) is also the generating function partially marked noncrossing trees with z marking vertices. So, we have that the formula that counts the plane trees described in this subsection with $n-1$ vertices is the same formula that enumerates partially marked noncrossing trees with n vertices.

4.3 Decorated ternary trees

Consider ternary trees (the edges are left-, middle- or right-edges) in which an edge emanating from a vertex of outdegree 1 comes in two colors if it is the right-edge. We call these trees as *decorated ternary trees.* In Figure 10, we have all the 19 decorated ternary trees on 3 vertices.

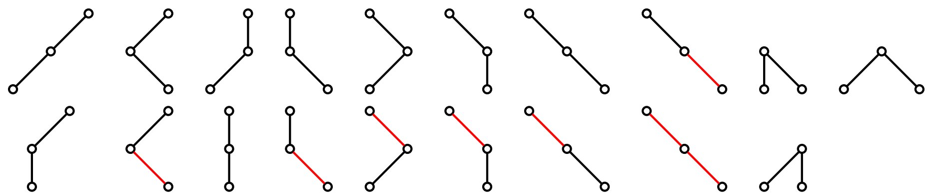

Figure 10: Decorated ternary trees on 3 vertices.

Let $T(z)$ be the generating function for these trees where z marks a vertex. There are three positions to attach subtrees (which could be empty) for each internal vertex. Each edge coming from a vertex of outdegree 1 is to come in two colors. The generating function is thus given by the functional equation

$$T(z) = z(1 + T(z))^3 + zT(z).$$

Again this is the generating function for partially marked noncrossing trees with z marking non-root vertices. So, we have that the formula for ternary trees described in this subsection with $n-1$ vertices also counts partially marked noncrossing trees on n vertices. The bijection between the set of decorated ternary trees and the labeled plane trees in the previous subsection can easily be seen.

5 Conclusion and future work

In this paper, we introduced a new combinatorial structure, which we termed partially marked noncrossing trees, and demonstrated that it is enumerated by a generalization of the classical formula for counting marked plane trees. We provided enumeration of these structures according to the number of vertices, marked edges, and leaves. The study was further extended to a broader class of generalized structures, which were likewise enumerated using the same parameters. We then explored connections between partially marked noncrossing trees and various known combinatorial structures, including noncrossing trees with multiple edges, plane trees in which some edges are labeled, and decorated ternary trees. Establishing explicit bijections between partially marked noncrossing trees and these structures, alongside other classical objects such as 2-plane trees, even trees, noncrossing partitions, Dyck paths, restricted lattice paths and Motzkin paths, presents an intriguing direction for future research. In particular, a promising avenue lies in extending this bijective approach to partially marked d-dimensional plane trees, potentially uncovering deeper combinatorial insights.

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