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Research Note

Eigenspaces for -2 in signed line graphs

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Abstract

It is known that -2 appears in the spectrum of a connected signed line graph if and only if its root is either (a) a balanced signed graph, not a tree, that spans a switching of the complete signed graph or (b) an unbalanced simply signed graph, not a unicyclic simply signed graph, that spans a switching of the signed doubled complete graph. In this paper, we compute the eigenspaces for -2 of these signed line graphs. The contribution can be seen as an alternative to [Discrete Appl. Math. 207 (2016) 29–38].

1 Introduction

A *signed graph* $\dot{G} = (G, \sigma)$ consists of an underlying graph $G = (V, E)$ with a *signature function* σ that maps the edge set E into $\{1, -1\}$. The edges mapped to 1 are *positive*, those mapped to -1 are *negative*, and together they comprise the edge set of $\dot{G}$. An ordinary graph is interpreted as a signed graph in which all edges are positive, i.e., with the all-positive signature; it is recognized in the text by the absence of a dot symbol.

If $\dot{G}$ has n vertices, then its *adjacency matrix* $A_{\dot{G}}$ is the $n \times n$ vertex-vertex $(0, 1, -1)$-matrix which is obtained from the adjacency matrix of its underlying graph by reversing the sign of all 1s which correspond to negative edges. By the *eigenvalues* and the *spectrum* of $\dot{G}$, we mean the eigenvalues and the spectrum of $A_{\dot{G}}$.

Signed graphs $\dot{G}$ and $\dot{H}$ are *switching equivalent* if $\dot{H}$ is obtained by selecting a vertex subset of $\dot{G}$ and reversing the sign of every edge with exactly one end in the selected subset. Switching equivalent signed graphs share the same spectrum.

In this paper we follow the definition of signed line graphs that can be found in [1, 5, 6, 7]. This concept is tailored to spectral graph theory and differs in sign from that of [8]; a detailed comparison is given in [2]. Accordingly, for $\dot{G}$ we introduce the vertex-edge orientation $\eta\colon V(G) \times E(G) \longrightarrow \{0, 1, -1\}$ formed by obeying the following rules: $\eta(i, jk) = 0$ if $i \notin$

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$\{j,k\}$, $\eta(i,ij) \in \{1,-1\}$ and $\eta(i,ij)\eta(j,ij) = -\sigma(ij)$. The vertex-edge incidence matrix B_η is the matrix whose (i,e) entry is $\eta(i,e)$. The adjacency matrix of a signed line graph $\mathcal{L}(\dot{G})$ is

$$A_{\mathcal{L}(\dot{G})} = B_\eta^\intercal B_\eta - 2I,$$

where I is the identity matrix. Simultaneously, $\dot{G}$ is referred to as a *root* of $\mathcal{L}(\dot{G})$. A signed line graph depends on the orientation η, but different orientations give switching equivalent signed line graphs. Also, switching equivalent signed graphs produce switching equivalent signed line graphs (see [2]).

A signed cycle is *positive* (resp. *negative*) if it switches (does not switch) to its underlying cycle. Equivalently, a signed cycle is positive (resp. negative) if the product of its edge signs is 1 (-1). A signed graph is *balanced* if it switches to its underlying graph; equivalently, it has no negative cycles. Otherwise, it is *unbalanced*.

In a signed graph, two parallel edges (i.e., two edges between the same pair of vertices) form a cycle of length 2 called a *digon*. A digon is positive if its edges have the same sign, and negative if they differ in sign. It follows that the existence of a positive digon in $\dot{G}$ implies the existence of parallel edges in its signed line graph. On the other hand, a negative digon produces non-adjacent vertices sharing the same neighborhood. A signed graph which allows parallel edges if and only if they form negative digons is called by Zaslavsky a *simply signed graph* [8]. Accordingly, $\mathcal{L}(\dot{G})$ has no multiple edges if and only if $\dot{G}$ is a simply signed graph. For a graph G, we denote by $\ddot{G}$ the *signed doubled graph* obtained from G by replacing each edge with a negative digon.

It is not difficult to see that the matrix $B_\eta^\intercal B_\eta$ is positive-semidefinite, and so the spectrum of every signed line graph is bounded from below by -2. Moreover, there is the following result; a starting point for this paper.

Theorem 1.1. *(cf. [4]) For a connected signed line graph $\mathcal{L}(\dot{G})$, the following statements are true:*

(i) $\dot{G}$ is a tree (with least eigenvalue > -2),

(ii) $\dot{G}$ is an unbalanced unicyclic simply signed graph (with least eigenvalue > -2),

(iii) $\dot{G}$ is balanced, not a tree, and up to switching spans the complete graph (with least eigenvalue -2),

(iv) $\dot{G}$ is unbalanced, not a unicyclic simply signed graph, and spans the signed doubled complete graph (with least eigenvalue -2).

Eigensbases for -2 in signed line graphs are computed by Belardo, Li Marzi and Simić in [1]. In this reference, the authors essentially deal with particular subgraphs of roots of signed line graphs, called *cores*, which are actually isomorphic to either simply signed theta graphs or simply signed dumbbells (both are defined in the next section). In this paper, we use the classification of Theorem 1.1 and consider the entire root, which requires a different proof technique. As a benefit, all foundations for an arbitrary signed line graph are determined; a foundation is also defined in the next section. Of course, the obtained eigenbases coincide, so in this view our contribution is just an alternative to [1]. It is also worth mentioning that the same eigenbases for ordinary graphs can be found in [3, Section 5.5]. However, the concept of signed line graphs does not generalize the concept of line graphs (see [2, 8]), and therefore the result obtained in this paper can be seen as an extension, but not a generalization, of the result for graphs.

For convenience, an edge lying between vertices i and j is denoted either by ij or $\{i, j\}$. The remaining notation and terminology are standard. For undefined notions, the reader is referred to any of [3, 4, 7].

The main results are formulated and proved in the next section. Section 3 contains some concluding remarks and examples.

2 Eigenbases for -2

For a connected signed graph $\dot{G}$ with m edges, let -2 be an eigenvalue of multiplicity $m-k$ in $\mathcal{L}(\dot{G})$. A spanning subgraph, say $\dot{H}$, of $\dot{G}$ with k edges such that -2 is not an eigenvalue of $\mathcal{L}(\dot{H})$ is known as a *foundation* for $\dot{G}$ [4]. A foundation is of particular interest since it gives rise to a maximal induced subgraph of $\mathcal{L}(\dot{G})$ without -2 in the spectrum. Also, we will see that foundations play a crucial role in determining the eigenspaces of -2 in signed line graphs. First, we prove the following result.

Theorem 2.1. *The following statements are true:*

(i) Let $\dot{G}$ be a connected balanced signed graph. A foundation for $\dot{G}$ is a spanning tree of $\dot{G}$.

(ii) Let $\dot{G}$ be a connected unbalanced simply signed graph. A foundation for $\dot{G}$ is a spanning subgraph in which each component is an unbalanced unicyclic simply signed graph.

Proof. If $\dot{H}$ is a foundation, then it is a maximal subgraph of $\dot{G}$ such that $\mathcal{L}(\dot{H})$ has no -2 as an eigenvalue.

We consider (i). By items (i) and (iii) of Theorem 1.1, $\dot{H}$ is acyclic. Moreover, it cannot be disconnected as by adding a link between two components we obtain a larger subgraph whose signed line graph has no -2 in the spectrum. Thus, $\dot{H}$ is a spanning tree.

For (ii), by items (ii) and (iv) of Theorem 1.1, $\dot{H}$ is a spanning subgraph in which each component is an unbalanced unicyclic simply signed graph. □

The forthcoming Theorems 2.3 and 2.6 are the main results of this paper, since they give the eigenbases for -2 in signed line graphs. Simultaneously, they ensure that the converse of the previous theorem is true: A subgraph of the type specified in items (i) and (ii) of Theorem 2.1 is a foundation for the corresponding signed graph $\dot{G}$.

We first consider signed line graphs of Theorem 1.1(iii). The following lemma is needed.

Lemma 2.2. *Let $\dot{C}$ be a positive signed cycle, and η a vertex-edge orientation on it. If the vertices of $\dot{C}$ are labelled by $0, 1, \ldots, k$ in the natural order, then a function $f\colon E(\dot{C}) \longrightarrow$*

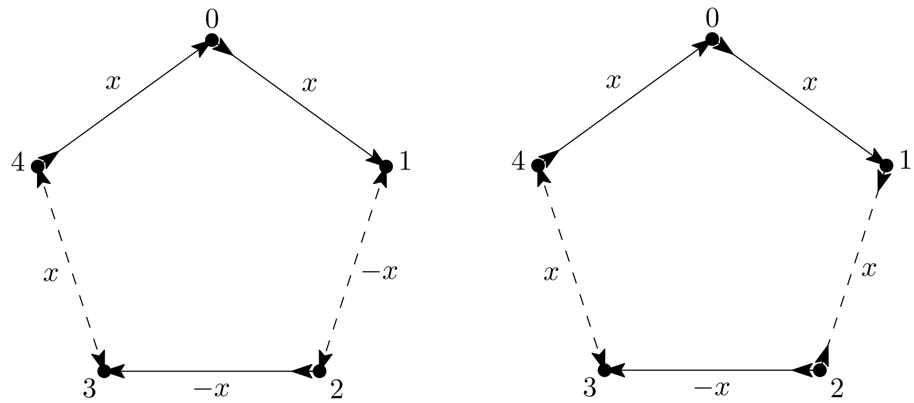

Figure 1: An illustration of Lemma 2.2. On the right-hand side, the orientation of the edge $\{1,2\}$ is changed. (Negative edges are dashed. An arrow pointing to a vertex indicates a positive vertex-edge orientation.)

$\{x,-x\}, x \in \mathbb{R}$, *defined by*

$$\begin{aligned} f(\{0,1\}) &= x,\\ f(\{i,i+1\}) &= -\eta(i,\{i-1,i\})\eta(i,\{i,i+1\})f(\{i-1,i\}),\ \textit{for } i := i \pmod{k+1}, \end{aligned}$$

is consistent in the sense that $-\eta(0,\{k,0\})\eta(0,\{0,1\})f(\{k,0\}) = f(\{0,1\})$.

Proof. Assume first that for every edge $\{i,i+1\}$, it holds:

$$\begin{aligned} \eta(i,\{i,i+1\}) &= -1, \quad \textit{if } \sigma(\{i,i+1\}) = 1,\\ \eta(i,\{i,i+1\}) &= 1, \quad\ \ \textit{if } \sigma(\{i,i+1\}) = -1. \end{aligned}$$

Observe that, in passing from $\{i-1,i\}$ to $\{i,i+1\}$, the function f changes the sign if and only if $\sigma(\{i,i+1\}) = -1$. Since $\dot{C}$ is positive, it has an even number of negative edges, which leads to the desired result. In changing the orientation on a single edge, f is negated on this edge and unchanged on the remaining edges, which completes the proof. □

The previous lemma is visualized in Figure 1.

For a signed graph $\dot{G}$, a fixed vertex-edge orientation η uniquely defines the signature σ on the edge set of the signed line graph $\mathcal{L}(\dot{G})$ in the following way: If u is a vertex incident to edges e and f of $\dot{G}$, then $\sigma(ef) = \eta(u,e)\eta(u,f)$.

Theorem 2.3. *Let $\dot{G}$ be a connected balanced signed graph distinct from a tree, $\dot{T}$ a spanning tree and $F = E(\dot{G}) \setminus E(\dot{T})$. For $e_0 \in F$, let $e_0, e_1, \ldots, e_k$ be the edges (listed in the natural order) of the unique cycle obtained by adding e_0 to $\dot{T}$. If σ is the signature on the edge set of $\mathcal{L}(\dot{G})$ defined by a fixed vertex-edge orientation on $\dot{G}$ and $m = |E(\dot{G})|$, then $\mathbf{x} \in \mathbb{R}^m$ that takes*

$$x_{e_i} = \begin{cases} 1, & i = 0,\\ -\sigma(e_{i-1}e_i)x_{e_{i-1}}, & 1 \le i \le k, \end{cases}$$

and zero on the remaining edges, is an eigenvector for -2 in $\mathcal{L}(\dot{G})$. Moreover, the $m - |E(\dot{T})|$ eiegnvectors formed in this way span the eigenspace for -2.

Proof. Let e simultaneously denote an edge of $\dot{G}$ and the associated vertex of $\mathcal{L}(\dot{G})$, and let x_e be the corresponding entry of $\mathbf{x}$. We denote the cycle given in the statement formulation by $\dot{C}$ and a fixed orientation by η. Observe that $\dot{C}$ is positive since $\dot{G}$ is balanced.

If e is non-adjacent to any edge of $\dot{C}$, then $x_e = 0$ and also $(A_{\mathcal{L}(\dot{G})}\mathbf{x})_e = 0$.

Suppose that $e \in \dot{C}$, say $e = e_i$. Let u (resp. v) denote the vertex between e_{i-1} and e_i (e_i and e_{i+1}). We have

$$\begin{aligned}(A_{\mathcal{L}(\dot{G})}\mathbf{x})_e &= \sigma(e_{i-1}e_i)x_{e_{i-1}} + \sigma(e_ie_{i+1})x_{e_{i+1}} = \eta(u, e_{i-1})\eta(u, e_i)x_{e_{i-1}} + \eta(v, e_i)\eta(v, e_{i+1})x_{e_{i+1}}\\ &= -(\eta(u, e_{i-1})\eta(u, e_i))^2 x_{e_i} - (\eta(v, e_i)\eta(v, e_{i+1}))^2 x_{e_i} = -2x_e,\end{aligned}$$

as desired. Lemma 2.2 ensures that the previous equality remains valid for $i \in \{0, k\}$.

Suppose that $e \notin \dot{C}$ and e is incident to a single vertex, say u, of $\dot{C}$. It holds $x_e = 0$. On the other hand, if u is located between e_i and e_{i+1}, then

$$\begin{aligned}(A_{\mathcal{L}(\dot{G})}\mathbf{x})_e &= \sigma(ee_i)x_{e_i} + \sigma(ee_{i+1})x_{e_{i+1}} = \eta(u, e)\eta(u, e_i)x_{e_i} + \eta(u, e)\eta(u, e_{i+1})x_{e_{i+1}}\\ &= \eta(u, e)\eta(u, e_i)x_{e_i} + \eta(u, e)\eta(u, e_{i+1})(-\eta(u, e_i)\eta(u, e_{i+1})x_{e_i}) = 0.\end{aligned}$$

The remaining case, in which e is not in $\dot{C}$ but lies between two vertices belonging to the same cycle, is considered with slight modifications in the previous part of the proof. Therefore, $\mathbf{x}$ is an eigenvector associated with -2.

The eiegnvectors constructed in this way are linearly independent because, for each $f \in F$, the f-entry is non-zero in exactly one of them. This completes the proof. □

An illustration of Theorem 2.3 is given in Figure 1, where x is fixed to 1.

We proceed with signed line graphs of Theorem 1.1(iv) and prove two lemmas.

Lemma 2.4. *Let $\dot{C}$ be a negative signed cycle and η a vertex-edge orientation on it, and let the vertices of $\dot{C}$ be labeled by $0, 1, \dots, k$ in the natural order. For $f\colon E(\dot{C}) \longrightarrow \{x, -x\}, x \in \mathbb{R}$, defined by*

$$f(\{0,1\}) = x,$$
$$f(\{i, i+1\}) = -\eta(i, \{i-1, i\})\eta(i, \{i, i+1\})f(\{i-1, i\}),\ \textit{for } 1 \le i \le k, i := i \pmod{k+1},$$

we have $\eta(0, \{k, 0\})\eta(0, \{0, 1\})f(\{k, 0\}) = f(\{0, 1\})$.

Proof. Assume that the orientation is assigned as in Lemma 2.2. Since $\dot{C}$ is negative, when passes the entire cycle f changes its sign an odd number of times, which leads to the desired result. The remainder of the proof (concerning different orientations) is analogous to the corresponding part of the proof of Lemma 2.2. □

For an illustration, one may see Figure 1, change the sign of $\{4, 0\}$ and set $f(\{4, 0\}) = -x$.

At this point we introduce two types of a simply signed bicyclic graph. The first one contains a simply signed theta graph as an induced subgraph, where a *simply signed theta graph* consists of 2 vertices joined by three edge-disjoint paths (since negative digons are allowed, at most two paths are of length 1). The second one contains a simply signed dumbbell as an induced subgraph, where a *simply signed dumbbell* consists of two cycles sharing at most one vertex. See Figure 2.

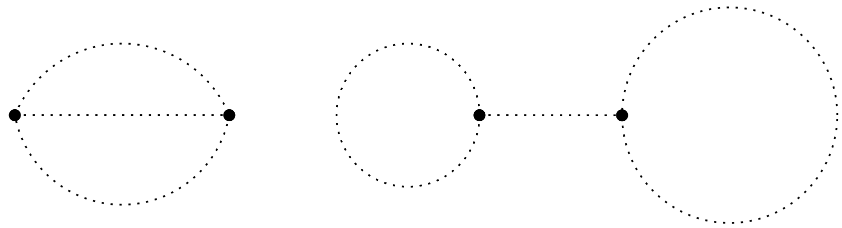

Figure 2: A sketch of a simply signed theta graph and a simply signed dumbbell. In the latter case, the specified vertices may be identified.

Lemma 2.5. *A simply signed theta graph contains either one or three positive cycles.*

Proof. Since there are three paths lying between two fixed vertices, the number of negative edges has the same parity for either one or all three pairs of paths. By combining them, we arrive at the desired result. □

We are ready to prove the following result.

Theorem 2.6. *Let $\dot{G}$ be a connected unbalanced simply signed graph distinct from a unicyclic simply signed graph, $\dot{U}$ a spanning subgraph whose components are unbalanced unicyclic simply signed graphs and $F = E(\dot{G}) \setminus E(\dot{U})$. Adding e ($e \in F$) to $\dot{U}$ creates either a single positive cycle or a simply signed dumbbell without positive cycles.*

Let σ be the signature on the edge set of $\mathcal{L}(\dot{G})$ defined by a fixed vertex-edge orientation on $\dot{G}$ and $m = |E(\dot{G})|$.

(i) If e creates a positive cycle, then $\mathbf{x} \in \mathbb{R}^m$ formed as in Theorem 2.3 is an eigenvector for -2 in $\mathcal{L}(\dot{G})$.

(ii) Suppose that e creates a simply signed dumbbell without positive cycles. Let $f_0, f_1, \ldots, f_\ell$ be edges in the path between two cycles, $e'_0, e'_1, \ldots, e'_{k_1}$ and $e''_0, e''_1, \ldots, e''_{k_2}$ edges of two cycles, all in the natural order, such that f_0 (resp. f_ℓ) is adjacent to e'_0, e'_{k_1} (e''_0, e''_{k_2}). Then $\mathbf{x} \in \mathbb{R}^m$ that takes

$$x_{f_i} = \begin{cases} 2, & i = 0, \\ -\sigma(f_{i-1}f_i)x_{f_{i-1}}, & 1 \leq i \leq \ell, \end{cases}$$

$$x_{e'_i} = \begin{cases} -\sigma(f_0 e'_i), & i \in \{0, k_1\}, \\ -\sigma(e'_{i-1}e'_i)x_{e'_{i-1}}, & 1 \leq i \leq k_1 - 1, \end{cases}$$

$$x_{e''_i} = \begin{cases} -\sigma(f_\ell e''_i)x_{f_\ell}/2, & i \in \{0, k_2\}, \\ -\sigma(e''_{i-1}e''_i)x_{e''_{i-1}}, & 1 \leq i \leq k_2 - 1, \end{cases}$$

and zero on the remaining edges is an eigenvector for -2 in $\mathcal{L}(\dot{G})$.

The $m - |E(\dot{U})|$ eiegnvectors formed in this way span the eigenspace for -2.

Proof. If e lies between two vertices of a fixed cycle, say $\dot{C}$, of $\dot{U}$, then e creates a simply signed theta graph. Since $\dot{C}$ is negative, by employing Lemma 2.5 we obtain that this theta graph has exactly one positive cycle.

If at most one vertex incident to e belongs to a cycle of $\dot{U}$, then either e lies between vertices of the same component of $\dot{U}$ or makes a link between two components. In the former case, e creates a simply signed theta graph or a simply signed dumbbell. As before, on the basis of Lemma 2.5, we deduce that the first situation results in a single positive cycle. Evidently, the second situation results in either a single positive cycle or a simply signed dumbbell without positive cycles. Finally, if e creates a link between two unicyclic components, then it creates a simply signed dumbbell. Since the components are unbalanced, the dumbbell has no positive cycles.

Now, (i) follows from Theorem 2.3, and we proceed with (ii). Let e denote an edge of $\dot{G}$ and the corresponding vertex of $\mathcal{L}(\dot{G})$, and let x_e be the corresponding entry of $\mathbf{x}$. The bicyclic component created by e is denoted by $\dot{B}$ and belonging cycles by $\dot{C}_1$ and $\dot{C}_2$. Both cycles are negative.

If e is non-adjacent to any edge of $\dot{B}$, then $x_e = (A_{\mathcal{L}(\dot{G})}\mathbf{x})_e = 0$. Further, for $\ell \geq 1$, we have

$$\begin{aligned}(A_{\mathcal{L}(\dot{G})}\mathbf{x})_{f_0} &= \sigma(f_0e'_0)x_{e'_0} + \sigma(f_0e'_{k_1})x_{e'_{k_1}} + \sigma(f_0f_1)x_{f_1}\\ &= -\sigma(f_0e'_0)^2 - \sigma(f_0e'_{k_1})^2 - 2\sigma(f_0f_1)^2 = -4 = -2x_{f_0}.\end{aligned}$$

Edges f_i $(0 < i < \ell)$, f_ℓ and f_0 (when $\ell = 0$) are considered in a similar way. Next, we have

$$\begin{aligned}(A_{\mathcal{L}(\dot{G})}\mathbf{x})_{e_0} &= \sigma(f_0e'_0)x_{f_0} + \sigma(e'_0e'_1)x_{e'_1} + \sigma(e'_0e'_{k_1})x_{e'_{k_1}} = 2\sigma(f_0e'_0) - \sigma(e'_0e'_1)^2x_{e'_0} + x_{e'_0}\\ &= -2x_{e'_0} - \sigma(e'_0e'_1)^2x_{e'_0} + x_{e'_0} = -2x_{e'_0},\end{aligned}$$

where $\sigma(e'_0e'_{k_1})x_{e'_{k_1}} = x_{e'_0}$ follows from Lemma 2.4. The remaining edges of cycles and the situation in which a cycle is a negative digon are considered in a similar way.

Finally, if u is a vertex of $\dot{B}$, then $\sum \eta(u, f)x_f = 0$ holds by the way of construction of $\mathbf{x}$, where the summation goes over all edges incident to u. This means that if e is not in $\dot{B}$ but is incident to at least one vertex of $\dot{B}$, then $x_e = (A_{\mathcal{L}(\dot{G})}\mathbf{x})_e = 0$. This completes the proof. □

An illustration is given in Figure 3.

3 Comments and examples

Theorem 2.3 says that the vectors that correspond to cycles of a fundamental cycle basis create a basis of $\mathcal{E}(-2)$. Since $\dot{G}$ of the same theorem is balanced, it switches to its underlying graph G. If G contains a Hamiltonian path, then by labeling its vertices by $0, 1, \ldots, n-1$ in the natural order and choosing the orientation η satisfying

$$\eta(i, ij) = \begin{cases} -1, & i = j-1,\\ 1, & i < j-1,\end{cases} \tag{3.1}$$

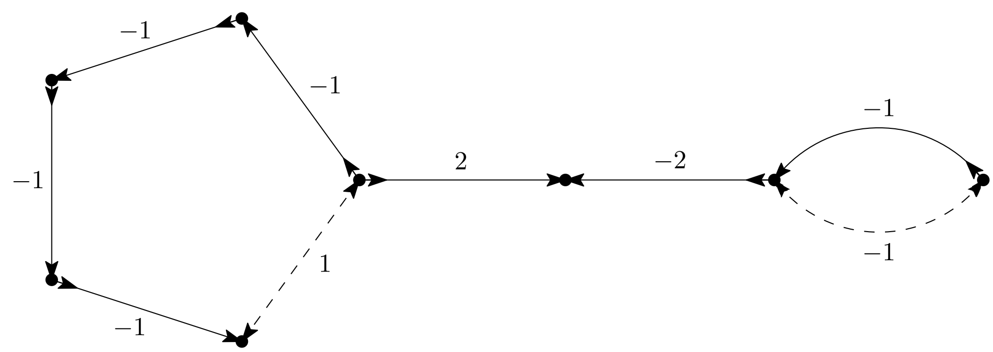

Figure 3: An illustration of Theorem 2.6(ii).

we obtain that all eigenvectors formed as in the theorem are non-negative. We recall from the first section that switching equivalent signed graphs give switching equivalent signed line graphs and different orientations again produce switching equivalent signed line graphs.

Example 3.1. Let P_n be a spanning path of the complete (unsigned) graph K_n. Suppose that the vertices of P_n are labeled in the natural order and η satisfies (3.1). Then the basis of $\mathcal{E}(-2)$ consists of the $\frac{(n-1)(n-2)}{2}$ vectors that take 1 on edges $\{i, i+1\}, \{i+1, i+2\}, \dots, \{i+k, i+k+1\}, \{i+k+1, i\}$, $i \geq 0, k \geq 1, i+k+1 \leq n-1$, and 0 otherwise.

Here is an illustration of an analogous situation for Theorem 2.6 when $\dot{G}$ contains a Hamiltonian cycle.

Example 3.2. Let $\ddot{C}_n$ be the signed doubled cycle with vertices $0, 1, \dots, n-1$. If n is odd, then $\ddot{C}_n$ contains a negative spanning cycle with the all-negative signature. For

$$\eta(i, ij) = \begin{cases} 1, & i = j-1, \sigma(ij) = -1, \\ -1, & i = j-1, \sigma(ij) = 1, \end{cases}$$

the basis of $\mathcal{E}(-2)$ consists of n vectors that take 1 on the positive edge $\{i, i+1\}, i \geq 0$, and alternate in sign on negative edges distinct from $\{i, i+1\}$ starting from -1 on $\{i+1, i+2\}$; vertex enumeration is mod n.

There is a slight modification if n is even, as a negative spanning cycle is not all-negative.

In other words, if $\dot{G}$ contains a negative spanning cycle, then the eigenbasis can be formed without dealing with simply signed dumbbells. Consequently, the entries of eigenvectors are in $\{0, 1, -1\}$.

The previous examples consider signed line graphs with exactly two (distinct) eigenvalues. We know from [7] that a signed line graph has two eigenvalues if and only if either it is a positive quadrangle, or its root is K_n or $\ddot{H}$, where H is a regular graph. In these cases, the eigenbasis for -2 gives the eigenbasis for the other eigenvalue; clearly, this is $\mathcal{E}(-2)^{\perp}$. For $L(\ddot{C}_n)$, the other eigenvalue is 2, and -2 and 2 are equal in multiplicity. Moreover, since $L(\ddot{C}_n)$ is bipartite, with an appropriate vertex labeling its adjacency matrix has the form

$\begin{pmatrix} O & B^\intercal \\ B & O \end{pmatrix}$. Then, if $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}$ is an eigenvector for -2, from

$$\begin{pmatrix} O & B^\intercal \\ B & O \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix} = \begin{pmatrix} -B^\intercal \mathbf{y} \\ B\mathbf{x} \end{pmatrix} = \begin{pmatrix} 2\mathbf{x} \\ -2\mathbf{y} \end{pmatrix} = 2 \begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix},$$

we see that $\begin{pmatrix} \mathbf{x} \\ -\mathbf{y} \end{pmatrix}$ is an eigenvector for 2. In other words, with the previous vertex labeling, $\mathcal{E}(-2)^\perp$ is spanned by vectors obtained by negating the second half of the vectors that span $\mathcal{E}(-2)$.

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