

# Enumeration of odd-dimensional partitions modulo 4

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## Abstract

The number of standard Young tableaux on a partition shape  $\lambda$  is called the dimension of the partition. Partitions with odd dimensions were enumerated by McKay and were further characterized by Macdonald in the 1970s. Let  $a_i(n)$  be the number of partitions of  $n$  with dimension congruent to  $i$  modulo 4. In this paper, we refine Macdonald’s and McKay’s results by computing  $a_1(n)$  and  $a_3(n)$  when  $n$  has no consecutive 1s in its binary expansion or when the sum of binary digits of  $n$  is 2.

## 1 Introduction

Representation theory offers a deep look into the inner workings of groups by studying their action on vector spaces. A vector space  $V$  is called a *representation* of  $G$  if the elements of  $G$  act like linear transformations on  $V$ . For a finite group, Maschke’s theorem tells us that any complex representation decomposes into a direct sum of  $G$ -invariant subspaces called *irreducible representations*, each indexed by a conjugacy class of  $G$ . The fact that the dimensions of irreducible representations are natural numbers leads one to question whether these numbers are counting some objects. An important family of objects of combinatorial consequence is *partitions of  $n$* . A partition of  $n$  is a weakly decreasing list of natural numbers that sum to  $n$ . Our focus in this paper is on the representation theory of  $\mathfrak{S}_n$ , the *symmetric group on  $n$  letters*, whose irreducible representations  $\text{Specht}(\lambda)$  are indexed by partitions  $\lambda$  of  $n$ . Every basis of  $\text{Specht}(\lambda)$  is indexed by *standard Young tableaux* (SYTs) on the shape  $\lambda$  (Prop. 2.5.9 and Thm. 2.6.4 in [21]). The number of SYTs on a shape can be computed using the well-known hook-length formula [6].

The direction of inquiry that motivates this paper is the modular behavior of the dimensions (also known as *degrees*) of the representations. Our story begins with the 2-page note of McKay [16] wherein he, inspired by Brauer’s assertion that “there may be important properties of character tables yet to be discovered”, computes the number of irreducible

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representations with odd dimensions for several groups. Let  $m_p(G)$  for a prime  $p$  and group  $G$  denote the number of irreducible representations whose dimensions of  $G$  are not divisible by  $p$ . The quantity  $m_2(G)$  is often a power of 2 in McKay’s computations, and takes a wonderfully concise form for the case of the symmetric group. If  $n$  is a natural number such that  $n = 2^{k_1} + \dots + 2^{k_\ell}$  with  $k_1 > k_2 > \dots > k_\ell$ , then  $m_2(\mathfrak{S}_n) = 2^{k_1+k_2+\dots+k_\ell}$ . Thus, all of the information about the enumeration of representations of  $\mathfrak{S}_n$  with odd dimensions is encoded in the binary expansion of  $n$ . The proof provided by McKay invokes the Murnaghan-Nakayama rule ([21, Thm. 10.4.2]), thereby reducing the problem to studying hooks of size a power of 2 on partition diagrams. In the same paper, McKay formulates the celebrated *McKay conjecture* which relates  $m_2(G)$  for a group  $G$  to  $m_2(N)$  for the normalizer  $N$  of a 2-Sylow subgroup of  $G$ . In his 1976 paper [19], Olsson discusses conditions that imply McKay’s conjecture for general primes  $p$  and for  $GL_q(n)$ . This conjecture was proved in [15] and a general form of the conjecture for all primes  $p$  was established in a recent paper [3] by Cabanes and Späth. Navarro, an important contributor to the proof of the McKay conjecture, said that the decades-long work on this problem has resulted in “beautiful, wonderful, deep mathematics.” [22]

We are interested in the quantity  $m_p(G)$  from a more enumerative and combinatorial perspective. The concise formula of McKay for  $m_2(n)$  was generalized to an arbitrary prime  $p$  through the use of *p-core towers* by Macdonald [13]. This proof method reduces the problem of finding the modularity of dimensions to the problem of labeling a  $p$ -ary tree with  $p$ -cores (Definition 1.1). From this starting point, many avenues of extension present themselves. In [14], Macdonald gave a formula for  $m_p(G)$  for Coxeter groups  $G$ . In [10], Huang extends these results to find the mod  $p$  properties of projective indecomposable modules of a deformation of the symmetric group called the 0-Hecke algebra. All these enumeration results are for primes  $p$ . For a general  $n$ , asymptotic results [20] show that  $m_n(\mathfrak{S}_N) = 0$  as  $N \rightarrow \infty$ .

Another direction of research, to which we contribute, tries to understand  $m_n(\mathfrak{S}_N)$  explicitly for a natural number  $n$ . The 1976 paper [19] by Olsson studies heights of characters using Macdonald’s approach of  $p$ -core towers which amounts to finding number of partitions with dimensions congruent to  $p^i \pmod{p^n}$  for  $n \geq i$ , and contains existence results (Prop. 3.9 in [19]) based on the relative size of the partition and its cores. From this proposition, we have an equivalent condition for the case  $p = 2$ , and thus an equivalent condition on the positivity of the number of partitions with dimensions  $2 \pmod{4}$ , but no explicit results. The explicit computations using the 2-core tower method can be found in [11]. In this paper, we deal with the remaining cases modulo 4. Define  $a_i(n)$  to be the number of partitions of  $n$  with dimensions congruent to  $i$  modulo 4. Amrutha P and T. Geetha tackle the problem of computing  $m_{2^k}(\mathfrak{S}_n)$  in their paper [1]. They provide general recursive results and find  $m_4(\mathfrak{S}_{2^\ell})$  and  $m_8(\mathfrak{S}_{2^\ell})$ , while also providing a characterization for partitions of  $2^\ell$  with dimensions congruent to 2 modulo 4.

The primary motivation for our project to compute  $a_1(n)$  and  $a_3(n)$  stems from the work [8] of Spallone and Ganguly on the characterization of spinorial representations of  $\mathfrak{S}_n$  and  $\mathfrak{A}_n$  (the group of even permutations of  $\{1, 2, \dots, n\}$ ). They prove that the spinorality of  $\text{Specht}(\lambda)$  is equivalent to the existence of a congruence condition modulo 8 involving the dimension of  $\text{Specht}(\lambda)$  and its value at a certain product of transpositions. As mentioned in the introduction to their paper, spinorality of Galois representations has applications to

number theory.

In this paper, we establish results for  $a_1(n)$  and  $a_3(n)$  for sparse numbers (Definition 4.1), which when combined with results for  $a_2(n)$  from [11] allows us to compute  $m_4(\mathfrak{S}_n)$  exactly for sparse numbers. Furthermore, as outlined in [12], these results can be used to find  $m_2(\mathfrak{A}_n)$ . Our attempt to solve the mod 4 problem for partition dimensions is a foray into understanding the modulo 8 properties of the dimension, and thus better understanding the spinorality of symmetric and alternating group representations. This problem is also of independent interest as it showcases the difficulty inherent in computing  $m_n(G)$  for general  $n$ . We only capture a sliver of this expansive enumeration endeavor, but we do so with a reinforcement of the same assertion that inspired McKay to investigate this field in the first place – “there are many properties of character tables yet to be discovered”.

### 1.1 Background

We split our discussion of the necessary concepts and definitions for this paper into two parts. The first is this section where we discuss the well-known combinatorial concepts that are needed to understand the main problem tackled in the paper. In Section 2, we present the definitions necessary to understand the main proof method. We start with the definition of a partition.

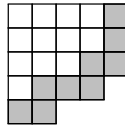
We call a  $k$ -tuple of positive integers  $\lambda = (\lambda_1, \dots, \lambda_k)$  a *partition* of size  $n$  if  $\lambda_1 + \lambda_2 + \dots + \lambda_k = n$ . We denote this by  $\lambda \vdash n$ . We denote the *length of the partition* by  $\ell(\lambda) = k$ . We construct the *Ferrers diagram* of  $\lambda$ , denoted by  $\mathcal{Y}(\lambda)$ , by placing  $\lambda_i$  top-left justified boxes in the  $i^{\text{th}}$  row from top. For  $\lambda = (3, 3, 2, 1)$ , we have

$$\mathcal{Y}(\lambda) = \begin{array}{|c|c|c|} \hline \square & \square & \square \\ \hline \square & \square & \square \\ \hline \square & \square & \\ \hline \square & & \\ \hline \end{array}$$

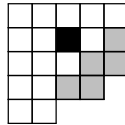
For  $\lambda \vdash n$ , we can fill the boxes of  $\mathcal{Y}(\lambda)$  with numbers in the set  $\{1, 2, \dots, n\}$ . A *standard Young tableau (SYT)* on the shape  $\lambda$  is a filling of  $\mathcal{Y}(\lambda)$  such that the entries increase strictly along rows (left to right) and along columns (top to bottom). Continuing with the above example, the following filling is an SYT on  $(3, 3, 2, 1)$ :

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 7 \\ \hline 3 & 5 & 8 \\ \hline 4 & 9 & \\ \hline 6 & & \\ \hline \end{array}$$

The number of SYTs on shape  $\lambda$  is denoted by  $f^\lambda$ . We call this quantity the *dimension of  $\lambda$*  and in the literature the terminology *degree of  $\lambda$*  also appears [16]. We can explicitly compute  $f^\lambda$  using the *hook-length formula* (Eq. (1.1)) which requires us to first define hook-lengths. We call the boxes in Ferrers diagrams *cells*. We label the cell in the  $i^{\text{th}}$  row from top and  $j^{\text{th}}$  column from the left by  $(i, j)$ . Write  $(i, j) \in \mathcal{Y}(\lambda)$  if  $1 \leq i \leq \ell(\lambda)$  and  $1 \leq j \leq \lambda_i$ . Define the *border of  $\lambda$*  to be the set of all cells  $(i, j) \in \mathcal{Y}(\lambda)$  such that at least one of  $(i, j + 1)$ ,  $(i + 1, j)$  or  $(i + 1, j + 1)$  is not contained in  $\mathcal{Y}(\lambda)$ . The partition  $(5, 5, 5, 4, 2)$  has the border shaded in gray as below:



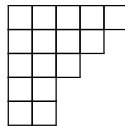
For a cell  $(i, j)$  in  $\mathcal{Y}(\lambda)$ , define the *hook*<sup>1</sup> of  $\lambda$  at  $(i, j)$  to be the set of border cells weakly below row  $i$ , and weakly to the right of column  $j$ . The number of cells contained in the hook at  $(i, j)$  is called its *hook-length* and is denoted by  $h_{i,j}$ . A hook of length  $t$  is called a  $t$ -rim hook. The partition  $(5, 5, 5, 4, 2)$  has a 5-rim hook (shaded in gray) at  $(2, 3)$  (shaded in black):



It is through these hook-lengths that the computation of  $f^\lambda$  is made possible. From [6], we know

$$f^\lambda = \frac{n!}{\prod_{(i,j) \in \mathcal{Y}(\lambda)} h_{i,j}}. \tag{1.1}$$

In Section 2.2, we will describe an equivalent formula due to Frobenius which will be more amenable to the methods of this paper. Our methods will require us to *remove* the hooks from partitions and see how the mod 4 behavior of  $f^\lambda$  changes. The Frobenius form is well-suited for such methods. As the process of hook removal and the output of this process are so crucial, we describe it now in terms of Ferrers diagrams, and later in terms of sets in Proposition 2.7. In terms of diagrams, we *remove a  $t$ -rim hook  $H$  from  $\lambda$*  by removing all cells of  $H$  from  $\mathcal{Y}(\lambda)$ . Removing the 5-rim hook at  $(2, 3)$  from  $(5, 5, 5, 4, 2)$  results in the partition  $(5, 4, 3, 2, 2)$  with the following diagram:



If the removal of a  $t$ -rim hook  $H$  from  $\lambda$  results in the partition  $\mu$ , then we write  $H = \lambda/\mu$ .

**Definition 1.1** (*t*-core). If  $\mathcal{Y}(\mu)$  does not contain any  $t$ -rim hook, then we call  $\mu$  a *t*-core.

Given a partition  $\lambda$ , successive removals of  $t$ -rim hooks result in a partition that is eventually a *t*-core. The *t*-core thus obtained is independent of the order in which  $t$ -rim hooks are removed [9, Thm. 2.7.16], and is called the *t*-core of  $\lambda$ . We denote it by  $\text{core}_t(\lambda)$ . The main proof method of this paper focuses on the mod 4 relationship between  $f^\lambda$  and  $f^\mu$  where  $\mu$  is the  $2^R$ -core of  $\lambda$ , and  $2^R$  is the largest power of 2 smaller than the size of  $\lambda$ .

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<sup>1</sup>In the literature, the objects we define are known as *rim hooks* while the terminology *hooks* is reserved for a related concept. The sets of hooks and rim hooks are in bijection, so we use the term “hooks” in place of “rim hooks” for convenience.

## 1.2 Main results

We denote the set of partitions of  $n$  with dimensions congruent to  $i \pmod 4$  by  $a_i(n)$  throughout this paper. The number of odd-dimensional partitions is denoted by  $a(n)$ , and by the work of McKay and Macdonald, we know that  $a(n) = 2^{k_1+k_2+\dots+k_l}$  where  $2^{k_i}$  appear in the binary expansion of  $n$ . In the notation of McKay described in the introduction,  $a(n) := m_2(\mathfrak{S}_n) = a_1(n) + a_3(n)$ . Furthermore,  $m_4(n) = a_1(n) + a_2(n) + a_3(n)$ . The problem of computing  $a_2(n)$  is tackled in [11] while in this paper we aim to find  $a_1(n)$  and  $a_3(n)$ . The computation of  $a_1(n)$  or  $a_3(n)$  is a hard question in general, and thus, we prove the results for sparse<sup>2</sup> numbers, which are positive integers with no consecutive ones in their binary expansions. For instance, 41 is sparse as its binary expansion is 101001.

The methods described in this paper compute the difference  $\delta(n) = a_1(n) - a_3(n)$  and the formulas for  $\delta(n)$  are more concise than the formulas for  $a_1(n)$  and  $a_3(n)$  themselves. We remark that the knowledge of  $\delta(n)$  and  $a(n)$  is sufficient to recover  $a_1(n)$  and  $a_3(n)$  as  $a_1(n) = \frac{a(n)+\delta(n)}{2}$  and  $a_3(n) = \frac{a(n)-\delta(n)}{2}$ , respectively. We now present the recursions and explicit computations, which are proved in this paper.

**Theorem 1.2.** *Let  $n, m, R \in \mathbb{N}$  with  $R \geq 2$  and  $m > 0$ . Suppose,  $n = 2^R + m$  with  $m < 2^{R-1}$ . Then we have*

$$\delta(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 4\delta(m), & \text{if } n \text{ is odd.} \end{cases}$$

Let  $\nu(n)$  denote the number of 1s in the binary expansion of  $n$ . For instance,  $\nu(41) = 3$ .

**Corollary 1.3.** *If  $n$  is a sparse number, then*

$$\delta(n) = \begin{cases} 2, & \text{if } n = 2 \\ 0, & \text{if } n > 2 \text{ is even} \\ 4^{\nu(n)-1}, & \text{if } n \text{ is odd.} \end{cases}$$

In the case where the binary expansion starts with two 1s followed by all zeroes, i.e.,  $n = 1100\dots$ , we have the following result.

**Theorem 1.4.** *Let  $n = 2^R + 2^{R-1}$  with  $R \geq 1$ . Then*

$$\delta(n) = \begin{cases} 2, & \text{if } R = 1 \\ 8, & \text{if } R = 2 \\ 0, & \text{otherwise.} \end{cases}$$

We prove Theorem 1.2 and Corollary 1.3 in Section 4, and Theorem 1.4 in Section 5. Before we get into the technical definitions and proofs, we give a broad overview of our proof ideas.

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<sup>2</sup>These numbers are called *fibbinary numbers* in the literature as they are enumerated by the Fibonacci numbers. The reader may refer to the OEIS entry A003714 [17].

### 1.3 Proof strategy and outline

Our final goal in this paper is to compute  $\delta(n)$  for non-negative integers  $n$ . Say we have a function  $\text{Od}$  on natural numbers which outputs  $+1$  when the input is  $1 \pmod 4$  and  $-1$  when the input is  $3 \pmod 4$ . This allows us to use  $\text{Od}(f^\lambda)$  to associate a sign  $+1$  to each partition with dimension  $1 \pmod 4$ , and  $-1$  to each partition with dimension  $3 \pmod 4$ . This process leaves us with  $a_1(n)$  partitions of  $n$  with associated sign  $+1$  and  $a_3(n)$  partitions of  $n$  with the sign  $-1$ . This allows us to write  $\delta(n) = a_1(n) - a_3(n)$  as  $\sum_\lambda \text{Od}(f^\lambda)$  where the sum is over all odd-dimensional partitions  $\lambda$ . The technique we adopt to compute this sum is recursion arising from the process of hook removal (Section 3). If  $n = 2^R + m$  with  $m < 2^R$ , then Macdonald’s work [13] tells us that each odd-dimensional partition  $\lambda$  of  $n$  corresponds to an odd-dimensional partition  $\mu$  of  $m$  obtained by removing a unique  $2^R$ -rim hook, or in other words,  $\mu$  is the  $2^R$ -core of  $\lambda$ . With this process of  $2^R$ -rim hook removal, we can associate a sign of  $\pm 1$  computed by the ratio  $\frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)}$ . Furthermore, define  $S(\mu)$  as the sum  $\sum_\lambda \frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)}$  over all odd-dimensional partitions  $\lambda$  of  $n$  that yield  $\mu$  after the  $2^R$ -rim hook removal process. When  $\text{Od}(f^\lambda)/\text{Od}(f^\mu)$  is  $+1$ , then the dimension of  $\lambda$  has the same residue modulo 4 as the dimension of  $\mu$ , and the residue is different when the ratio is  $-1$ . Thus, the quantity  $S(\mu)$  captures the difference between the number of partitions  $\lambda$  whose dimensions have the same or different residue modulo 4 compared to  $f^\mu$ . This allows us to write  $\delta(n) = \sum_\lambda \text{Od}(f^\lambda)$  with the sum over all odd-dimensional partitions  $\lambda$  of  $n$  as  $\sum_\mu S(\mu) \text{Od}(f^\mu)$  where the sum is now over all odd-dimensional partitions of  $m$ . The bulk of our paper will be devoted to understanding  $S(\mu)$  and  $\frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)}$  for odd-dimensional partitions  $\mu$  of  $m$  and  $\lambda$  of  $n$  as above. For a general  $n$ , this problem is difficult, as the value of  $S(\mu)$  can depend on the parts of  $\mu$  (Section 7). In this paper, we opt to study the special cases where  $n = 2^R + m$  with  $m < 2^{R-1}$  (Section 4), and the case where  $n = 2^R + 2^{R-1}$  for  $R \geq 1$  (Section 5). The usefulness of these specializations becomes evident in the fact that  $S(\mu)$  for  $\mu \vdash m$  or  $\mu \vdash 2^{R-1}$  is independent of the partition  $\mu$  and only depends on the parity of  $m$  (Theorem 4.11). In these cases, we may factor out some constant  $s_m$  dependent only on  $m$  from the sum, allowing us to construct a very simple recursion for  $\delta(n)$  which is  $\delta(n) = s_m \sum_\mu \text{Od}(f^\mu) = s_m \delta(m)$  where the sum is over all odd-dimensional  $\mu$  of  $m$ .

The implementation of this proof strategy will require technical arguments relating to objects arising from partitions called  $\beta$ -sets. In Section 2, we introduce these objects, their effectiveness in describing hook removal, and the Frobenius hook-length formula that uses these to compute the dimension. We also present the characterization of odd-dimensional partitions due to Macdonald [13] in terms of  $\beta$ -sets and introduce the notion of  $t$ -parents. This section is also where we properly introduce the function  $\text{Od}$  and lay out some of its required properties. In Section 3, we prove the modular hook recursion that establishes  $\text{Od}(f^\lambda)/\text{Od}(f^\mu)$  for partitions  $\lambda$  of  $n = 2^R + m$  and  $\mu$  of  $m$ . In Section 4, we focus on the case of sparse numbers, and establish the recursion relating  $\delta(n)$  and  $\delta(m)$  (Theorem 1.2) where  $m < 2^{R-1}$ . In Section 5, we tackle the simplest non-sparse case where  $n = 2^R + 2^{R-1}$  to prove Theorem 1.4. The proof in this section resembles the proofs presented in Section 4, but as we shall see, the casework required in this case is indicative of the general difficulty of the modulo 4 problem as seen through the lens of our methods. In Section 6, we provide

the interested reader with some pointers that might be helpful in resolving the general cases, and the obstructions that one may face. In Section 7, we provide data for  $\delta(n)$  for  $n < 128$  and comment on some patterns within it. At the beginning of each subsection, we provide some exposition to motivate our methods and situate the reader as the proofs are technical and computationally involved.

We summarize the specializations considered in this paper. Throughout, we write  $n = 2^R + m$  for  $0 \leq m < 2^R$ , where  $2^R$  extracts the largest power of 2 smaller than  $n$ , and  $m$  is what remains after subtraction. In Section 2 and Section 3, we consider the general case of  $m < 2^R$  as all our propositions hold without any restrictions. In Section 4, we specialize to the case  $m < 2^{R-1}$  which allows for explicit recursion formulas and enumeration. In Section 5, we take the first step beyond the non-sparse case by considering  $m = 2^{R-1}$ . In Section 7, we discuss strategies for the case where  $2^{R-1} < m < 2^R$ .

## 2 Development of the modular hook recursion formula

The main recursion underlying our methods requires some background to set up. In this section, we present an overview of the foundation needed to derive the formula in the next section and understand the proof methods in the rest of the paper. The definitions hold for all values of  $n = 2^R + m$ ,  $m < 2^R$ , and we will in later sections impose further restrictions on  $m$ .

Let  $n = 2^R + m$  for natural numbers  $n$ ,  $R$  and  $m$  with  $m < 2^R$ . To compute  $\delta(n)$ , we study the modulo 4 properties of the Frobenius hook-length formula (Proposition 2.9) which counts the number of SYTs  $f^\lambda$  of a given shape  $\lambda$ . The main ingredient of this formula is the set of hook-lengths which appear in the cells of the first column of the diagram (Notation 2.1). For an odd-dimensional partition  $\lambda$ , we keep track of the residue modulo 4 of  $f^\lambda$  through the function  $\text{Od}(f^\lambda)$  (Section 2.3) which outputs  $+1$  when  $f^\lambda$  has residue  $1 \pmod 4$  and  $-1$  when  $f^\lambda$  has residue  $3 \pmod 4$ . To establish the main recursion in Section 3, we explore how the removal of a  $2^R$ -rim hook from a diagram  $\mathcal{Y}(\lambda)$  affects the value of  $\text{Od}(f^\lambda)$ . The process of hook-removal can be understood naturally in terms of  $\beta$ -sets, which motivates our usage of them as the central object in our paper.

### 2.1 $\beta$ -sets

The Frobenius formula, described in the next section, that we will be using to compute dimensions takes as input the first column hook-lengths of a partition. In fact, the formula works for a general class of equivalent objects called  $\beta$ -sets which we now describe. These  $\beta$ -sets are a natural candidate for our purposes as removal of a hook on diagrams can be understood as a simple alteration of a single element in a  $\beta$ -set as described in Proposition 2.7.

**Notation 2.1.** The set of first column hook-lengths of  $\lambda$  is denoted by  $H(\lambda) := \{h_{i,1} \mid 1 \leq i \leq \ell(\lambda)\}$ , and the elements are listed in descending order. Explicitly, if  $\lambda = (\lambda_1, \dots, \lambda_k)$ , then  $h_{i,1} = \lambda_i + k - i$ . In this paper, we often drop the column subscript 1 and just write  $h_i$  for  $h_{i,1}$ .

For any finite  $X \subset \mathbb{Z}_{\geq 0}$ , define the  $r$ -shift of  $X$  to be

$$X^{+r} = \{x + r \mid x \in X\} \cup \{0, \dots, r - 1\}.$$

Also, define  $X^{+0} = X$ . Thus, an  $r$ -shift of  $X$  increases each element of  $X$  by  $r$  and inserts the elements  $0, 1, \dots, r - 1$ . For a partition  $\lambda$ , sets of the form  $H(\lambda)^{+r}$  for  $r \geq 0$  are known as the  $\beta$ -sets of  $\lambda$ . We now present some important properties of  $\beta$ -sets as remarks.

**Remark 2.2.** For any weakly decreasing  $r$ -tuple of non-negative integers  $w$ , we define  $H(w) = \{w_i + r - i : 1 \leq i \leq r\}$ . An alternate way to view the  $\beta$ -set  $H(\lambda)^{+r}$  is by using partitions with trailing zeroes. If  $r \geq \ell(\lambda)$ , then let  $\rho$  be the weakly decreasing list of integers obtained by appending  $r - \ell(\lambda)$  zeroes to the end of  $\lambda$ . If we compute  $H(\rho) = \{h_1, \dots, h_r\}$  using the formula  $h_i = \rho_i + r - i$ , we obtain exactly  $H(\lambda)^{+r - \ell(\lambda)}$ . For instance, let  $\lambda = (4, 3, 3, 1)$  with  $H(\lambda) = \{7, 5, 4, 1\}$ . If  $r = 6$ , then  $H(\lambda)^{+2} = \{9, 7, 6, 3, 1, 0\}$ . On the other hand, we have  $\rho = (4, 3, 3, 1, 0, 0)$  and we find that  $H(\rho)$  agrees with  $H(\lambda)^{+2}$ .

**Remark 2.3.** As the largest hook-length can at most encompass all the cells of a partition, the maximum element of  $H(\lambda)$  is at most  $|\lambda|$ . In general, the maximum element of an  $r$ -shift of  $\lambda$  is at most  $|\lambda| + r$ .

**Remark 2.4.** We can reconstruct the original partition from a  $\beta$ -set with  $k$  elements by setting  $\lambda_i = h_i + i - k$  and considering only the values where  $\lambda_i > 0$ . For instance, if we have a  $\beta$ -set  $X = \{8, 4, 3, 1, 0\}$  of length  $k = 5$ , then the corresponding partition  $\lambda$  has non-negative parts  $\lambda_1 = 8 - 5 + 1 = 4$ ,  $\lambda_2 = 4 - 5 + 2 = 1$ ,  $\lambda_3 = 3 - 5 + 3 = 1$ ,  $\lambda_4 = 1 - 5 + 4 = 0$  and  $\lambda_5 = 0 - 5 + 5 = 0$ . Thus,  $\lambda = (4, 1, 1)$  and  $X$  is  $H(\lambda)^{+2}$ .

From the above remark, we see that if two sets can be expressed as  $r$ -shifts of each other, then they must arise as first column hook-length sets of the same partition. This motivates the following equivalence relation.

**Notation 2.5.** For finite sets  $X, Y \subset \mathbb{N}$ , we say  $X \sim_{\beta} Y$  if and only if  $X = Y^{+r}$  or  $Y = X^{+r}$  for some  $r \in \mathbb{Z}_{\geq 0}$ .

We now explicitly describe how to interpret  $t$ -rim hook removal and addition in terms of  $\beta$ -sets. Given a set  $S$  with  $a \in S$  and  $b \notin S$ , define  $S[a \rightarrow b]$  to be set obtained from  $S$  where  $a$  is replaced by  $b$ . In notation,  $S[a \rightarrow b] = (S \cup \{b\}) \setminus \{a\}$ . In the cases where  $b$  is already in  $S$  or  $a$  is not in  $S$ , we define  $S[a \rightarrow b] = S$ .

**Lemma 2.6.** For a set  $S \subset \mathbb{Z}_{\geq 0}$  with  $a \in S$  and  $b \in \mathbb{Z}_{\geq 0}$  which may or may not be in  $S$ , we have  $S[a \rightarrow b][b \rightarrow a] = S$ .

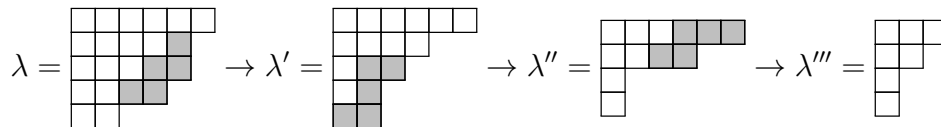
*Proof.* If  $b \notin S$ , then  $S[a \rightarrow b]$  contains all the same elements as  $S$  except with  $a$  replaced by  $b$ . The set  $S[a \rightarrow b][b \rightarrow a]$  replaces the  $b$  back by  $a$ , giving us  $S$ . On the other hand, if  $b \in S$ , then  $S[a \rightarrow b] = S$  and  $S[b \rightarrow a] = S$ .  $\square$

**Proposition 2.7.** Let  $\lambda$  be a partition and  $X$  be a  $\beta$ -set of  $\lambda$ . The diagram  $\mathcal{Y}(\lambda)$  contains a  $t$ -rim hook if and only if there exists an  $h$  in  $X$  such that  $h \geq t$  and  $h - t$  is not in  $X$ . Furthermore, if  $\mu$  is the partition obtained after removing the  $t$ -rim hook, then  $H(\mu) \sim_{\beta} H(\lambda)[h \rightarrow h - t]$ .

*Proof.* See Corollary 1.5 on page 7 of [18] or Lemma 2.7.1 of [9]. □

Proposition 2.7 states that the removal of a  $t$ -rim hook from the diagram of  $\lambda$  is expressed in terms of the  $\beta$ -set  $X$  of  $\lambda$  by selecting an element  $h \in X$  and reducing it by  $t$  (if  $h - t$  is not already an element of  $X$ ). Equivalently, one may add a  $t$ -rim hook to  $\mathcal{Y}(\lambda)$  by selecting an element  $h'$  in a sufficiently large  $\beta$ -set  $Y$  of  $\lambda$  and increasing it by  $t$  (if  $h' + t$  is not in  $X$ ). In the case where no value  $h$  in  $X$  can be reduced by  $t$  as  $h - t$  already occurs in  $X$ , we declare the partition  $\lambda$  to be a  $t$ -core.

**Example 2.8.** In this example, we find the 5-core of  $\lambda = (6, 5, 5, 4, 2)$  using the process of hook removal described on  $\beta$ -sets, and then display the same process using diagrams. We have  $H(\lambda) = \{10, 8, 7, 5, 2\}$ . In this case, we have that  $8 \in H(\lambda)$  but  $3 \notin H(\lambda)$ . Thus, there exists a 5-rim hook in  $\mathcal{Y}(\lambda)$  which we can remove to obtain  $\lambda'$  (not to be confused with the transpose) such that  $H(\lambda') = \{10, 7, 5, 3, 2\}$ . Now, we can further replace 5 by 0 to obtain  $H(\lambda'') = \{10, 7, 3, 2, 0\} \sim_{\beta} \{9, 6, 2, 1\}$ . Finally, we can replace 9 by 4 to get  $H(\lambda''') = \{6, 4, 2, 1\}$ . No further replacements can be performed. Thus,  $\lambda''' = (3, 2, 1, 1)$  is the 5-core of  $\lambda$ . This can be visualized using Ferrers diagrams with the 5-rim hooks removed at each step shaded in gray:



## 2.2 Characterization of odd-dimensional partitions

The following proposition is due to Frobenius:

**Proposition 2.9** (Eq. 6 in [7]). *Let  $\lambda$  be a partition of  $n$  and let  $X = \{x_1, \dots, x_k\}$  with  $x_1 > \dots > x_k$  be any  $\beta$ -set of  $\lambda$ . Then*

$$f^\lambda = \frac{n! \prod_{1 \leq i < j \leq k} (x_i - x_j)}{\prod_{i=1}^k x_i!}.$$

Note that in the above proposition  $X$  is any  $\beta$ -set, and not necessarily the set of first-column hook-lengths, and can possibly contain a 0.

**Remark 2.10.** The above expression for  $f^\lambda$  can be split into three parts: the factorial of the size of the partition,  $n!$ ; the product of the factorials of the elements of the  $\beta$ -set,  $\prod_i x_i!$ ; and the Vandermonde product,  $\text{Van}(x_1, x_2, \dots, x_k) := \prod_{1 \leq i < j \leq k} (x_i - x_j)$ .

**Example 2.11.** Let  $\lambda = (4, 4, 2, 1, 1)$  with  $\beta$ -set  $X = \{10, 9, 6, 4, 3, 1, 0\}$ . Then

$$f^\lambda = \frac{12! \cdot (10 - 9)(10 - 6)(10 - 4) \dots (3 - 1)(3 - 0)(1 - 0)}{10! 9! 6! 4! 3! 1! 0!} = 4455.$$

We call  $\lambda$  an *odd-dimensional partition* if  $f^\lambda$  is odd. For instance, the partition  $(4, 4, 2, 1, 1)$  in Example 2.11 is odd-dimensional. We now restate the characterization of odd-dimensional partitions by Macdonald [13] in terms of hooks and cores.

**Proposition 2.12.** *If  $\lambda$  is a partition of  $n = 2^R + m$  with  $m < 2^R$ , then  $\lambda$  is an odd-dimensional partition if and only if  $\lambda$  contains exactly one  $2^R$ -rim hook and  $\text{core}_{2^R}(\lambda)$  is also an odd-dimensional partition.*

*Proof.* This proposition is stated as Lemma 1 in [2] and a proof of it is given in Section 6 of the same paper. □

**Remark 2.13.** A consequence of the above proposition is that  $\lambda$  is an odd-dimensional partition of  $2^R$  if and only if  $\lambda$  is a  $2^R$ -rim hook.

The repeated application of the above proposition tells us that if we can remove a unique hook of size the largest power of 2 smaller than the size of the partition at each step, then the partition is odd-dimensional. For instance, if  $\lambda$  is a partition of  $2^6 + 2^3 + 2^1$ , we can remove a unique  $2^6$ -rim hook from  $\lambda$  to yield  $\mu$ , a unique  $2^3$ -rim hook from  $\mu$  to yield  $\nu$ , and a unique 2-rim hook from  $\nu$  (which is  $\nu$  itself by the above remark) to get  $\emptyset$ . As might be evident from this example, binary expansions play an important role in our techniques, so we present the notation for them that is used throughout the text.

**Notation 2.14** (Binary expansion). Let  $n = \sum_{i=0}^k b_i 2^i$  with  $b_i \in \{0, 1\}$  such that  $b_k = 1$ . We call the  $b_i$  the *bits* of  $n$ . We use the shorthand  $n = [b_k \dots b_0]_2$  in place of the expansion.

Each odd-dimensional partition can be constructed by repeated addition of larger hooks corresponding to each positive bit in its binary expansion starting from the empty shape  $\emptyset$ . In the above example, we removed hooks to get to the empty partition, but in our proofs, we start from a partition, and then add hooks in all possible ways. We call the partitions obtained by addition of  $t$ -rim hooks to  $\mathcal{Y}(\mu)$  the  $t$ -parents of  $\mu$ .

**Definition 2.15.** Let  $\lambda$  be an odd-dimensional partition of  $n = 2^R + m$  with  $m < 2^R$ . If  $\text{core}_{2^R}(\lambda) = \mu$ , then we call  $\lambda$  a  $2^R$ -parent of  $\mu$ .

Definition 2.15 is unambiguous as  $\lambda$  contains a unique  $2^R$ -rim hook which when removed results in a partition of size  $m < 2^R$  which is a  $2^R$ -core. Equivalently,  $\lambda$  is a  $2^R$ -parent of  $\mu$  if  $\lambda$  is obtained by the addition of a single  $2^R$ -rim hook from  $\mu$ . To understand how the  $\beta$ -sets of  $2^R$ -parents can be derived from the  $\beta$ -sets of the starting partition, we present the next proposition. Recall from Proposition 2.7 that the addition (resp. the removal) of a hook to  $\mu$  corresponds to increasing (resp. decreasing) an element in the  $\beta$ -set of  $\mu$ . We can create  $\beta$ -sets of  $2^R$ -parents of an odd-dimensional partition  $\mu$  by starting out with a  $\beta$ -set  $B$  of  $\mu$  and replacing some element  $b$  of  $B$  by  $b + 2^R$ .

We split the parents into two categories depending on the  $\beta$ -set we choose. Type I  $2^R$ -parents are formed by choosing the  $\beta$ -set  $H(\mu)$  and replacing some  $h \in H(\mu)$  by  $h + 2^R$ , whereas Type II  $2^R$ -parents arise from replacing 0 by  $2^R$  in  $\beta$ -sets which are  $r$ -shifts of  $H(\mu)$  for  $1 \leq r \leq 2^R$ . This is explicitly stated in the following proposition.

**Proposition 2.16.** *If  $\mu \vdash m < 2^R$  is obtained from  $\lambda \vdash m + 2^R$  after removal of a  $2^R$ -rim hook, then exactly one of the following holds:*

1. (Type I)  $H(\lambda) = H(\mu)[x \rightarrow x + 2^R]$  for some  $x \in H(\mu)$ ,

2. (Type II)  $H(\lambda) = H(\mu)^{+r}[0 \rightarrow 2^R]$  for  $1 \leq r \leq 2^R$  such that  $2^R \notin H(\mu)^{+r}$ .

*Proof.* By Proposition 2.7 and Proposition 2.12, recall that the  $\beta$ -set obtained after a removal of a  $2^R$ -rim hook from  $H(\lambda)$  results in a  $\beta$ -set of  $\mu$ . In the notation from Notation 2.5, we have  $H(\lambda)[h \rightarrow h - 2^R] \sim_\beta H(\mu)$ . As  $\lambda$  is formed by addition of a hook to  $\mu$ ,  $\lambda$  contains at least as many parts as  $\mu$ , which means  $\ell(\lambda) \geq \ell(\mu)$ . This turns the previous equivalence into the equality  $H(\lambda)[h \rightarrow h - 2^R] = H(\mu)^{+r}$  where  $r \geq 0$ . As explained in Remark 2.17 below, the range of  $r$  is  $\{0, 1, 2, \dots, 2^R\}$ . When  $r = 0$ , we get  $H(\lambda)[h \rightarrow h - 2^R] = H(\mu)$  where  $h$  is an element of  $H(\lambda)$  and  $h - 2^R$  is an element of  $H(\mu)$ . If we let  $h - 2^R = x$ , we obtain the equality  $H(\lambda)[x + 2^R \rightarrow x] = H(\mu)$  for  $x \in H(\mu)$ . By replacing  $x$  by  $x + 2^R$  on both sides, we get

$$H(\lambda)[x + 2^R \rightarrow x][x \rightarrow x + 2^R] = H(\mu)[x \rightarrow x + 2^R]$$

which is equal to  $H(\lambda)$  by Lemma 2.6. Following Remark 2.17 below, we may take  $h - 2^R = 0$  without loss of generality. Thus, we have for some  $0 < r \leq 2^R$ ,  $H(\lambda)[2^R \rightarrow 0] = H(\mu)^{+r}$ , which by another application of Lemma 2.6 gives the desired result. We emphasize that as  $2^R$  is being replaced in  $H(\lambda)$  by 0,  $2^R$  cannot be an element in  $H(\mu)^{+r}$ , thus we get no corresponding  $2^R$ -parents for the  $r_h$ -shift where  $r_h = 2^R - h$  for some  $h \in H(\mu)$ , leading to the condition in part 2.  $\square$

**Remark 2.17.** In Proposition 2.16, there are two quirks in the definition of Type II parents. Why does the  $r$ -shift only range from 1 to  $2^R$ ? This is because any  $\beta$ -set  $Y = H(\mu)^{+r}$  with  $r > 2^R$  would already contain the element  $2^R$ , thus  $Y[0 \rightarrow 2^R] = Y$  would not result in a  $2^R$ -parent. Second, why are we not considering the more general case of replacing any  $h \in H(\mu)^{+r}$  with  $h + 2^R$ , and only focusing on replacing 0 with  $2^R$ ? It follows from the definitions that the  $\beta$ -set  $H(\mu)^{+r}[h \rightarrow h + 2^R]$  is equal to  $(H(\mu)^{r-h}[0 \rightarrow 2^R])^{+h}$  which makes it equivalent to  $H(\mu)^{r-h}[0 \rightarrow 2^R]$ , and thus not contributing a new  $2^R$ -parent. The definition of replacing only the 0 by  $2^R$  in the shifted  $\beta$ -sets ensures that each  $2^R$ -parent is counted exactly once.

**Remark 2.18.** In [5], the Type II  $t$ -parents arise from the addition of *special*  $t$ -rim hooks, which are  $t$ -rim hooks whose southwestern-most cell lies in the first (leftmost) column.

**Example 2.19.** Let  $\mu = (4, 2)$  be an odd-dimensional partition and let  $2^R = 8$ . We have  $H(\mu) = \{5, 2\}$ . We construct the Type I 8-parent  $\lambda^{(2)} = (9, 5)$  corresponding to  $x = 2$  by adding 8 to the entry  $2 \in H(\mu)$  and obtain with  $H(\lambda^{(2)}) = \{10, 5\}$ . We construct a Type II parent  $\nu = (4, 4, 3, 1, 1, 1)$  by first finding the 4-shift of  $H(\mu)$ ,  $H(\mu)^{+4} = \{9, 6, 3, 2, 1, 0\}$ , and replacing 0 by 8 to obtain  $H(\nu) = \{9, 8, 6, 3, 2, 1\}$ . We list out the Type I 8-parents of  $\mu = (4, 2)$ :

- $\lambda^{(1)} = (12, 2)$  with  $H(\lambda^{(1)}) = \{13, 2\}$  arising by adding 8 to the element 5.
- $\lambda^{(2)} = (9, 5)$  with  $H(\lambda^{(2)}) = \{10, 5\}$  arising by adding 8 to the element 2.

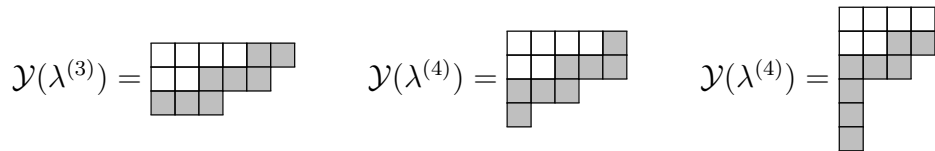
The corresponding Ferrers diagrams are shown below:

$$\mathcal{Y}(\lambda^{(1)}) = \begin{array}{|c|c|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & & & & & & & & & \\ \hline \end{array} \quad \mathcal{Y}(\lambda^{(2)}) = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}.$$

We list out the Type II 8-parents of  $\mu = (4, 2)$ :

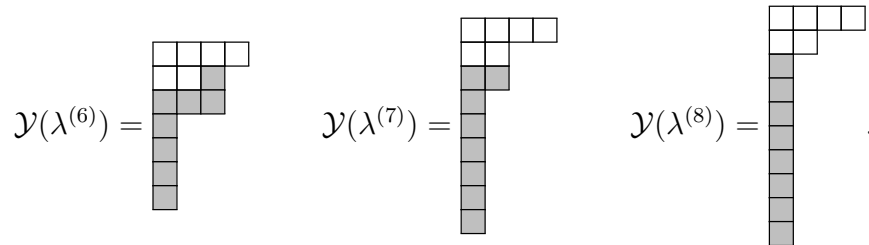
- $\lambda^{(3)} = (6, 5, 3)$  with  $H(\lambda^{(3)}) = \{8, 6, 3\}$  arising from a 1-shift of  $H(\mu)$ , and replacing 0 by 8 in  $H(\mu)^{+1}$ .
- $\lambda^{(4)} = (5, 5, 3, 1)$  with  $H(\lambda^{(4)}) = \{8, 7, 4, 1\}$  arising from a 2-shift of  $H(\mu)$ , and replacing 0 by 8 in  $H(\mu)^{+2}$ .
- $\lambda^{(5)} = (4, 4, 3, 1, 1, 1)$  with  $H(\lambda^{(5)}) = \{9, 8, 6, 3, 2, 1\}$  arising from a 4-shift of  $H(\mu)$ , and replacing 0 by 8 in  $H(\mu)^{+4}$ .

The corresponding Ferrers diagrams are shown below:



- $\lambda^{(6)} = (4, 3, 3, 1, 1, 1, 1)$  with  $H(\lambda^{(6)}) = \{10, 8, 7, 4, 3, 2, 1\}$  arising from a 5-shift of  $H(\mu)$ , and replacing 0 by 8 in  $H(\mu)^{+5}$ .
- $\lambda^{(7)} = (4, 2, 2, 1, 1, 1, 1, 1, 1)$  with  $H(\lambda^{(7)}) = \{12, 9, 8, 6, 5, 4, 3, 2, 1\}$  arising from a 7-shift of  $H(\mu)$ , and replacing 0 by 8 in  $H(\mu)^{+7}$ .
- $\lambda^{(8)} = (4, 2, 1, 1, 1, 1, 1, 1, 1, 1)$  with  $H(\lambda^{(8)}) = \{13, 10, 8, 7, 6, 5, 4, 3, 2, 1\}$  arising from an 8-shift of  $H(\mu)$ , and replacing 0 by 8 in  $H(\mu)^{+8}$ .

The corresponding Ferrers diagrams are shown below:



We note that there are no 8-parents arising from the shifts  $r_5 = 8 - 5 = 3$  and  $r_2 = 8 - 2 = 6$  as both  $H(\mu)^{+3} = \{8, 5, 2, 1, 0\}$  and  $H(\mu)^{+6} = \{11, 8, 5, 4, 3, 2, 1, 0\}$  contain both 8 and 0 simultaneously.

Proposition 2.16 can be used to deduce the counts of Type I and Type II  $2^R$ -parents. Note that there are exactly  $\ell(\mu)$  Type I  $2^R$ -parents as they all arise by choosing a certain element of  $H(\mu)$  to be replaced. On the other hand, the values of  $r$  for which  $H(\mu)^{+r}$  contains  $2^R$  (equivalently, have **no** corresponding  $2^R$ -parents) are in correspondence with the entries of  $H(\mu)$  by letting  $r^h = 2^R - h$  for any  $h \in H(\mu)$ . Thus, there are exactly  $2^R - \ell(\mu)$  values of  $r$  in the range  $1 \leq r \leq 2^R$  that lead to a  $2^R$ -parent of  $\mu$ . So, by adding the two contributions, we see that any odd-dimensional partition  $\mu$  has exactly  $2^R$   $2^R$ -parents. Iterating this method recovers the count  $a(n)$  of odd-dimensional partitions due to McKay [16] and Macdonald [13].

**Proposition 2.20.** *Let  $n = 2^{k_1} + \dots + 2^{k_r}$  with  $k_1 > \dots > k_r$ . Then the number of odd-dimensional partitions of  $n$  is given by  $a(n) := 2^{k_1 + \dots + k_r}$ .*

**Example 2.21.** If  $n = 74 = 2^6 + 2^3 + 2^1$ , then the unique odd-dimensional partition of 0 has 2 odd-dimensional 2-parents (Remark 2.13). Any odd-dimensional partition of 2 has 8  $2^3$ -parents giving us a total of 16 odd-dimensional partitions. Then any odd-dimensional partition of 16 has 64  $2^6$ -parents, giving us a total of  $2^{6+3+1} = 1024$  odd-dimensional partitions of 74.

### 2.3 Od function

To compute  $\delta(n)$ , we consider all odd-dimensional partitions and then study how they refine further modulo 4. With each partition with dimension 1 modulo 4, we wish to associate the value 1, and with each partition with dimension 3 modulo 4, we wish to associate the value  $-1$ . The sum of these values over all odd-dimensional partitions of  $n$  is  $\delta(n)$ . We extract these values by using the function Od.

**Definition 2.22.** Let  $n$  be a natural number. We define  $\text{Od}(n) \in \{-1, 1\}$  by finding the largest odd number  $k$  dividing  $n$ . If  $k \equiv 1 \pmod{4}$ , then  $\text{Od}(n) = 1$ , else if  $k \equiv -1 \pmod{4}$ , then  $\text{Od}(n) = -1$ .

**Example 2.23.** Let  $n = 1088$  which factors as  $64 \times 17$ . As 17 is the largest number dividing  $n$  and  $17 \equiv 1 \pmod{4}$ , we have  $\text{Od}(1088) = 1$ .

When  $f^\lambda$  is odd, it is often a result of cancellation of even factors among the three parts (Remark 2.10) of the Frobenius hook-length formula. We do not consider the factors modulo 4 directly, but instead opt to first extract the odd part of the formula and consider its value modulo 4. This method works as  $\{-1, 1\}$  forms a multiplicative group in  $\mathbb{Z}_4$ .

**Lemma 2.24.** For all  $m, n \geq 1$ ,  $\text{Od}(mn) = \text{Od}(m) \text{Od}(n)$ . When  $n$  divides  $k$ ,  $\text{Od}(k/n) = \text{Od}(k) / \text{Od}(n)$ .

*Proof.* The largest odd number dividing  $mn$  is the product of the largest odd numbers dividing  $m$  and  $n$  respectively. Considering this equality modulo 4 gives us the result. The latter result follows by letting  $k = mn$ .  $\square$

**Example 2.25.** Let  $m = 36$  and  $n = 42$ . Then the odd part of  $m$  is 9 while the odd part of  $n$  is 7. The odd part of  $mn = 1512$  is 63. So, we have  $\text{Od}(9) \text{Od}(7) = 1 \cdot -1 = -1 = \text{Od}(63)$ .

**Remark 2.26.** Let  $n$  have the binary expansion  $[b_k b_{k-1} \dots b_0]_2$ . Let  $j$  be the minimum index such that  $b_j = 1$ . Then,  $\text{Od}(n) \equiv [b_{j+1} b_j]_2 \pmod{4}$ . Equivalently, one may view the process of finding  $\text{Od}(n)$  as the binary expansion of  $n$ , removing all the rightmost zeros, and returning the rightmost two digits of the newly obtained string mod 4.

## 3 The modular hook recursion

Our goal in this section is to derive a relationship between the values of  $\text{Od}(f^\lambda)$  and  $\text{Od}(f^\mu)$  where  $\lambda$  is the  $2^R$ -parent of  $\mu$ . As Od is multiplicative (Lemma 2.24) and  $\text{Od}(n)$  is its own reciprocal for every  $n$ , it would be sufficient for us to consider how the terms in the Frobenius hook-length formula (Proposition 2.9) change in passage from  $f^\lambda$  to  $f^\mu$ . As two of the three parts (Remark 2.10) of the Frobenius hook-length formula contain factorials, we first analyze how factorials behave under Od.

### 3.1 Odd part of factorial modulo 4

We derive an explicit formula for  $\text{Od}(n!)$  for a non-negative integer  $n$ . This formula depends on two statistics associated with binary expansions. The first one is  $\nu(n)$ , which computes the sum of bits of the binary expansion of  $n$ , or equivalently counts the number of 1s in the binary expansion of  $n$ . The other is a statistic  $D(n)$  defined in [4] that deals with pairs of consecutive bits in the binary expansion of  $n$ .

**Definition 3.1.** For a non-negative integer  $n$ , define  $D(n)$  to be the number of pairs of consecutive 1s in the binary expansion of  $n$ . Letting  $n$  have the binary expansion  $[b_k \dots b_0]_2$ ,  $D(n) = |\{0 \leq i \leq k - 1 : b_i \cdot b_{i+1} = 1\}|$ .

**Example 3.2.** We have  $7 = [111]_2$ ,  $10 = [1010]_2$  and  $112 = [1101111]_2$ . We compute  $D(7) = 2$ ,  $D(10) = 0$  and  $D(112) = 4$ .

We first derive a recursion for the odd part of  $n!$  modulo 4. The idea of the proof is to separately analyze the product of odd numbers and even numbers less than  $n$ , and build the recursion using the fact that  $\text{Od}(2k) = \text{Od}(k)$  for all  $k \geq 0$ .

**Lemma 3.3.** For any natural number  $n$ ,

$$\text{Od}(n!) = (-1)^{\lfloor (n+1)/4 \rfloor} \text{Od}(\lfloor n/2 \rfloor!).$$

where  $\lfloor n \rfloor$  denotes the integral part of  $n$ .

*Proof.* By the multiplicativity of  $\text{Od}$  (Lemma 2.24), we have

$$\text{Od}(n!) = \prod_{r=1}^n \text{Od}(r) = \prod_{\substack{1 \leq r \leq n \\ r \text{ odd}}} \text{Od}(r) \prod_{\substack{1 \leq r \leq n \\ r \text{ even}}} \text{Od}(r).$$

As multiplication by 2 does not affect the odd part, we have  $\text{Od}(2r) = \text{Od}(r)$  which gives

$$\prod_{\substack{1 \leq r \leq n \\ r \text{ even}}} \text{Od}(r) = \prod_{k=1}^{\lfloor n/2 \rfloor} \text{Od}(2k) = \prod_{k=1}^{\lfloor n/2 \rfloor} \text{Od}(k) = \text{Od}(\lfloor n/2 \rfloor!).$$

We can split the product over odd numbers depending on their residue modulo 4 as

$$\begin{aligned} \prod_{\substack{1 \leq r \leq n \\ r \text{ odd}}} \text{Od}(r) &= \prod_{\substack{1 \leq r \leq n \\ r \in 4\mathbb{N}+1}} \text{Od}(r) \prod_{\substack{1 \leq r \leq n \\ r \in 4\mathbb{N}-1}} \text{Od}(r) \\ &= 1 \cdot \prod_{\substack{1 \leq r \leq n \\ r \in 4\mathbb{N}-1}} (-1). \end{aligned}$$

As there are exactly  $\lfloor (n+1)/4 \rfloor$  terms of the form  $4x - 1$  less than or equal to  $n$  where  $x \in \mathbb{N}$ , we get

$$\text{Od}(n!) = (-1)^{\lfloor (n+1)/4 \rfloor} \text{Od}(\lfloor n/2 \rfloor!).$$

□

By iterating the above recursion, we can find an explicit formula for  $\text{Od}(n!)$ .

**Lemma 3.4.** *For any natural number  $n$ , we have*

$$\text{Od}(n!) = (-1)^{D(n)+\nu(\lfloor n/4 \rfloor)},$$

where  $\lfloor n \rfloor$  denotes the integral part of  $n$ .

*Proof.* Let  $n = [b_k \dots b_0]_2$ . In this proof we will omit the brackets and the 2 in the subscript of binary notation, and simply write  $b_k b_{k-1} \dots b_0$ . Taking the integral part of a natural number after division by 2 has the effect of removing the rightmost bit from the binary expansion of  $n$ , which in notation is  $\lfloor \frac{b_\ell \dots b_0}{2} \rfloor = b_\ell \dots b_1$ . By Lemma 3.3, we get

$$\text{Od}([b_k b_{k-1} \dots b_0]_2!) = (-1)^{\lfloor \frac{b_k \dots b_0 + 1}{4} \rfloor} \text{Od}([b_k b_{k-1} \dots b_1]_2!)$$

We now apply the recursion on the  $\text{Od}$  term on the right to get

$$\text{Od}([b_k b_{k-1} \dots b_0]_2!) = (-1)^{\lfloor \frac{b_k \dots b_0 + 1}{4} \rfloor + \lfloor \frac{b_k \dots b_1 + 1}{4} \rfloor} \text{Od}([b_k b_{k-1} \dots b_2]_2!).$$

Continuing in this manner, we have

$$\text{Od}(n!) = (-1)^{\lfloor \frac{b_k \dots b_0 + 1}{4} \rfloor + \lfloor \frac{b_k \dots b_1 + 1}{4} \rfloor + \lfloor \frac{b_k \dots b_2 + 1}{4} \rfloor + \dots + \lfloor \frac{b_k b_{k-1} + 1}{4} \rfloor} \text{Od}(b_k!).$$

Now we look at the sum in the exponent. For any natural number  $m$ , we interpret taking the integral part of  $m/4$  as chopping off the *two* rightmost bits in the binary expansion of  $m$ . Suppose the two rightmost bits of  $m$  are  $b_j b_{j-1}$ , then if  $0 \leq [b_j b_{j-1}]_2 < 3$ , adding 1 to  $m$  will only affect the two rightmost bits, thus  $\lfloor (m+1)/4 \rfloor = \lfloor m/4 \rfloor$ . On the other hand, if  $[b_j b_{j-1}]_2 = 3$ , or equivalently, both  $b_j$  and  $b_{j-1}$  are 1, we have  $\lfloor (m+1)/4 \rfloor = \lfloor m/4 \rfloor + 1$ . From this we deduce that each time a consecutive pair of 1s appears in the binary expansion of  $n$ , we get a term of 1 added in the exponent. Thus,

$$\begin{aligned} & \left\lfloor \frac{b_k \dots b_0 + 1}{4} \right\rfloor + \left\lfloor \frac{b_k \dots b_1 + 1}{4} \right\rfloor + \left\lfloor \frac{b_k \dots b_2 + 1}{4} \right\rfloor + \dots + \left\lfloor \frac{b_k b_{k-1} + 1}{4} \right\rfloor \\ &= \left\lfloor \frac{b_k \dots b_0}{4} \right\rfloor + \left\lfloor \frac{b_k \dots b_1}{4} \right\rfloor + \dots + \left\lfloor \frac{b_k b_{k-1} b_{k-2}}{4} \right\rfloor + \left\lfloor \frac{b_k b_{k-1}}{4} \right\rfloor + D(n). \end{aligned}$$

Plugging this in the exponent yields

$$\text{Od}(n!) = (-1)^{D(n)} (-1)^{\lfloor \frac{b_k \dots b_0}{4} \rfloor + \lfloor \frac{b_k \dots b_1}{4} \rfloor + \dots + \lfloor \frac{b_k b_{k-1} b_{k-2}}{4} \rfloor + \lfloor \frac{b_k b_{k-1}}{4} \rfloor} \text{Od}(b_k!).$$

Using our interpretation of  $\lfloor m/4 \rfloor$  as removing the two rightmost bits in the binary expansion of  $n$ , we may write the exponent as:

$$\text{Od}(n!) = (-1)^{D(n)} \cdot (-1)^{b_k b_{k-1} \dots b_2 + b_k b_{k-1} \dots b_3 + \dots + b_k + 0} \text{Od}(b_k!).$$

As  $b_k \in \{0, 1\}$ ,  $b_k! = 1$ , and thus  $\text{Od}(b_k!) = 1$ . So,

$$\text{Od}(n!) = (-1)^{D(n)} (-1)^{b_k \dots b_2 + b_k \dots b_3 + \dots + b_k + 0}.$$

The value of a natural number  $m$  raised to  $-1$  only depends on the parity of  $m$ , or in notation,  $(-1)^{b_k \dots b_j} = (-1)^{b_j}$ . Using this, we obtain,

$$\text{Od}(n!) = (-1)^{D(n) + b_2 + b_3 + \dots + b_k} = (-1)^{D(n) + \nu(\lfloor n/4 \rfloor)}. \quad \square$$

### 3.2 The modular hook recursion

In this section, we study the ratio between  $\text{Od}(f^\lambda)$  and  $\text{Od}(f^\mu)$  when  $\lambda$  is a  $2^R$ -parent of  $\mu$ . As we have seen, the removal of a  $2^R$ -rim hook from the diagram of  $\lambda$  to obtain  $\mu$  corresponds to decrementing the entry  $h$  of  $H(\lambda)$  so as to obtain a  $\beta$ -set of  $\mu$ . We call the entry  $h$  we decrease the *affected hook-length*.

**Definition 3.5.** For partitions  $\mu$  and  $\lambda$  such that  $\lambda$  is the  $2^R$ -parent of  $\mu$ , define the *affected hook-length*  $h_\mu^\lambda$  to be the element of  $H(\lambda)$  such that replacing  $h_\mu^\lambda$  by  $h_\mu^\lambda - 2^R$  yields a  $\beta$ -set of  $\mu$ . In our hook removal notation,  $H(\mu) \sim_\beta H(\lambda)[h_\mu^\lambda \rightarrow h_\mu^\lambda - 2^R]$ .

We see later that the ratio  $\text{Od}(f^\lambda)/\text{Od}(f^\mu)$  for  $\mu \vdash m$  and its  $2^R$ -parent  $\lambda$  depends on three main factors: the size of  $\lambda$ , the affected hook-length, and a statistic  $\eta_\mu^\lambda$ , which is computed (with certain caveats) by looking at the number of elements we “jump over” when sending  $h_\mu^\lambda$  to  $h_\mu^\lambda - 2^R$ . Let  $H(\lambda) = \{h_1, h_2, \dots, h_k\}$  with the entries ordered as  $h_1 > h_2 > \dots > h_k$ . Then there is a  $\beta$ -set  $B$  of  $\mu$  containing all the same entries except some  $h_j = h_\mu^\lambda$  is replaced by  $h_j - 2^R$ . To motivate the following definition, we emphasize that the only elements that differ between the products  $\prod_{\substack{i < j \\ h_i, h_j \in H(\lambda)}} (h_i - h_j)$  and  $\prod_{\substack{i < j \\ h'_i, h'_j \in B}} (h'_i - h'_j)$ , are

the ones involving  $h_\mu^\lambda$ .

**Definition 3.6.** Let  $\lambda$  be a  $2^R$ -parent of  $\mu$  with  $|\mu| < 2^R$ . Let  $h_\mu^\lambda \in H(\lambda)$  be the affected hook-length (Definition 3.5). We define  $\eta_\mu^\lambda$  as follows:

$$\eta_\mu^\lambda = \left\{ x \in H(\lambda) \setminus \{h_\mu^\lambda\} : \frac{\text{Od}(|h_\mu^\lambda - x|)}{\text{Od}(|h_\mu^\lambda - 2^R - x|)} = -1 \right\}.$$

The above definition counts the number of  $-1$ s that appear in the product

$$\prod_{\substack{x \in H(\lambda) \\ x \neq h_\mu^\lambda}} \frac{\text{Od}(|h_\mu^\lambda - x|)}{\text{Od}(|h_\mu^\lambda - 2^R - x|)},$$

and thus this product is equal to  $(-1)^{\eta_\mu^\lambda}$ . In the next subsection, we compute the value of  $\eta_\mu^\lambda$  explicitly by comparing the odd parts modulo 4 of  $|h_\mu^\lambda - x|$  and  $|h_\mu^\lambda - 2^R - x|$ . The absolute values are necessary as the elements of a  $\beta$ -set are always ordered in descending order, and the difference between  $x$  and  $h_\mu^\lambda$  (or  $h_\mu^\lambda - 2^R$ ) must always be positive. To present the modular hook recursion, we need the next statistic on binary strings.

**Definition 3.7.** We define  $s_2(n)$  to be the sum of the leftmost two bits in the binary expansion of  $n$ . In other words, if  $n$  has the binary expansion  $[b_k b_{k-1} \dots b_0]_2$ , define  $s_2(n)$  as  $b_k + b_{k-1}$ .

We are now ready to present the result that forms the crux of our enumeration endeavor. Let  $B$  be the  $\beta$ -set of  $\mu$  obtained from  $H(\lambda)$  by decreasing the affected hook-length by  $2^R$ . We study the effect of  $\text{Od}$  on the three parts (Remark 2.10) in the product arising from  $f^\lambda/f^\mu$ . We see that  $\text{Od}(n!)/\text{Od}(m!)$  yields  $(-1)^{s_2(n)}$ . The product  $\prod_{h \in H(\lambda)} h!$  and  $\prod_{h' \in B} h'!$

differ in exactly one factor, and their ratio is  $h_\mu^\lambda!/(h_\mu^\lambda - 2^R)!$ . We see that this ratio yields the term  $(-1)^{s_2(h_\mu^\lambda)}$ . Finally, we look at the ratio of  $\text{Od}$  acting on the Vandermonde products arising from  $H(\lambda)$  and  $B$ , which by definition and our discussion above, yields  $(-1)^{\eta_\mu^\lambda}$ .

**Remark 3.8.** The following proof of Proposition 3.9 showcases the usefulness of our choice of the Frobenius hook-length formula (Proposition 2.9) over the one by Frame-Robinson-Thrall (Eq. (1.1)), as we are able to deal with  $\beta$ -sets (or equivalently, partitions with trailing zeros) and do not have to find the set of *all* hook-lengths for the  $2^R$ -core.

In the following proposition we consider  $n \geq 4$  as the removal of the leftmost bit of  $n$  would change the sum of bits for  $\lfloor n/4 \rfloor$  only when  $n \geq 4$ . This bound is not too concerning as our recursion is mostly applicable for large values of  $n$ , and the smaller values can be dealt with by hand (or considered initial values for the recursion).

**Proposition 3.9** (The modular hook recursion). *Let  $n \geq 4$  be a natural number such that  $n = 2^R + m$  with  $m < 2^R$ . Let  $\mu \vdash m$  and  $\lambda \vdash n$  be a  $2^R$ -parent of  $\mu$ . Then*

$$\text{Od}(f^\lambda) = (-1)^{s_2(n)+s_2(h_\mu^\lambda)+\eta_\mu^\lambda} \text{Od}(f^\mu).$$

*Proof.* By applying  $\text{Od}$  on the formula for  $f^\lambda$  in Proposition 2.9, we get

$$\text{Od}(f^\lambda) = (-1)^{D(n)+\nu(\lfloor \frac{n}{4} \rfloor)} \prod_{h \in H(\lambda)} (-1)^{D(h)+\nu(\lfloor \frac{h}{4} \rfloor)} \prod_{\substack{x > y \\ x, y \in H(\lambda)}} \text{Od}(x - y).$$

Choose a  $\beta$ -set  $B$  of  $\mu$  such that  $|B| = \ell(\lambda)$ . Then we obtain

$$\text{Od}(f^\mu) = (-1)^{D(m)+\nu(\lfloor \frac{m}{4} \rfloor)} \prod_{h' \in B} (-1)^{D(h')+\nu(\lfloor \frac{h'}{4} \rfloor)} \prod_{\substack{x' > y' \\ x', y' \in B}} \text{Od}(x' - y').$$

We first look at the ratio  $\frac{(-1)^{D(n)+\nu(\lfloor \frac{n}{4} \rfloor)}}{(-1)^{D(m)+\nu(\lfloor \frac{m}{4} \rfloor)}} = (-1)^{D(n)+\nu(\lfloor \frac{n}{4} \rfloor)-D(m)-\nu(\lfloor \frac{m}{4} \rfloor)}$ . Let  $n$  have the binary expansion  $[b_R \dots b_0]_2$ . The binary expansion of  $m = n - 2^R$  is obtained by removing the leftmost bit, and thus,  $m = [b_{R-1} \dots b_0]_2$ . If  $b_{R-1} = 1$ , then we lose one consecutive pair of 1s in the binary expansions from  $n$  to  $m$ , while on the other hand, if  $b_{R-1} = 0$ , the total number of consecutive pairs of 1s remains unchanged. It follows that  $D(n) - D(m) = b_{R-1}$ . For the digit sum, we get that removing a 1 from the left decreases the sum by 1, and as  $n \geq 4$ , the leftmost entry must be at least the third entry from the right. Thus,  $\nu(\lfloor n/4 \rfloor) - \nu(\lfloor m/4 \rfloor) = 1$ . These two facts give us

$$(-1)^{D(n)+\nu(\lfloor \frac{n}{4} \rfloor)-D(m)-\nu(\lfloor \frac{m}{4} \rfloor)} = (-1)^{1+b_{R-1}} = (-1)^{s_2(n)}.$$

We have that  $h_\mu^\lambda$  is in  $H(\lambda)$  but not in  $B$  whereas  $h_\mu^\lambda - 2^R$  is in  $B$  but not in  $H(\lambda)$ . All the other elements of the sets  $H(\lambda)$  and  $B$  are same. As  $n \geq 4$ , we have  $h_\mu^\lambda \geq 4$ , and so a similar computation as above yields

$$\frac{\prod_{h \in H(\lambda)} (-1)^{D(h)+\nu(\lfloor \frac{h}{4} \rfloor)}}{\prod_{h \in B} (-1)^{D(h)+\nu(\lfloor \frac{h}{4} \rfloor)}}$$

$$\begin{aligned}
 &= (-1)^{D(h_\mu^\lambda) + \nu(\lfloor \frac{h_\mu^\lambda}{4} \rfloor) - D(h_\mu^\lambda - 2^R) - \nu(\lfloor \frac{h_\mu^\lambda - 2^R}{4} \rfloor)} \\
 &= (-1)^{s_2(h_\mu^\lambda)}.
 \end{aligned}$$

Finally, we are left with the ratio of products

$$\frac{\prod_{\substack{x > y \\ x, y \in B}} \text{Od}(x - y)}{\prod_{\substack{x' > y' \\ x', y' \in B}} \text{Od}(x' - y')}$$

where all the differences not involving  $h_\mu^\lambda$  cancel out. This leaves us with the product

$$\prod_{\substack{x \in H(\lambda) \\ x \neq h_\mu^\lambda}} \frac{\text{Od}(|h_\mu^\lambda - x|)}{\text{Od}(|h_\mu^\lambda - 2^R - x|)}$$

which is exactly  $(-1)^{\eta_\mu^\lambda}$ . Combining all these three results gives us the desired recursion.  $\square$

### 3.3 Understanding $\eta_\mu^\lambda$

Although the recursion in Proposition 3.9 is quite compact, it must be untangled to aid in enumeration. The  $s_2$  terms can be computed directly from the binary expansions of  $n$  and  $h_\mu^\lambda$ , so we must investigate the  $\eta_\mu^\lambda$  term. When an entry  $h \in H(\lambda)$  is replaced by  $h - 2^R$ , we visualize this as  $h$  “jumping over” the entries between  $h$  and  $h - 2^R$  and each of these jumps may switch the sign of  $\text{Od}$ . We record the number of jumps in  $N_\lambda(h)$  (Notation 3.11). Some jumps result in a sign change, as in the case where  $6, 9 \in H(\lambda)$  and  $H(\mu) = H(\lambda)[9 \rightarrow 1]$ . Here,  $\text{Od}(9 - 6) = -\text{Od}(6 - 1)$ . However, not all jumps result in a sign change. For instance, let  $4, 8 \in H(\lambda)$  and  $H(\mu) \sim_\beta H(\lambda)[8 \rightarrow 0]$ . In this case,  $\text{Od}(8 - 4) = \text{Od}(4 - 0)$  which means that jumping over 4 does not change the sign of  $\text{Od}$ . This phenomenon makes analyzing  $\eta_\mu^\lambda$  difficult and makes the general problem of computing  $\delta(n)$  hard.

**Definition 3.10** (Indicator function). Denote by  $\mathbb{I}_X(x)$ , the *indicator function on  $X$* , which is equal to 1 if  $x \in X$  and equal to 0 otherwise.

The following statistic records the number of entries of  $H(\lambda)$  that occur between  $h_\mu^\lambda$  and  $h_\mu^\lambda - 2^R$ .

**Notation 3.11.** For a partition  $\lambda$  of  $n = 2^R + m$  with  $m < 2^R$  and  $h \in H(\lambda)$ , let  $N_\lambda(h) = |\{y \in H(\lambda) : h - 2^R < y < h\}|$ .

**Remark 3.12.** In terms of hooks on diagrams,  $N_\lambda(h_\mu^\lambda)$  is one less than the number of rows occupied by the hook  $\lambda/\mu$ . In the literature, this is often called the *height* of the hook. For instance, if  $H(\lambda) = \{11, 10, 7, 6, 3, 1\}$  and we obtain  $H(\mu) = \{11, 7, 6, 3, 2, 1\}$  by replacing 10 by 2, then the corresponding 8-rim hook (in gray) is given by

$$\lambda/\mu = (6, 6, 4, 4, 2, 1)/(6, 3, 3, 1, 1, 1) =$$

We see that 10 jumps over 7, 6, and 3, which makes  $N_\lambda(10) = 3$ , which is exactly one less than the number of rows occupied by the hook.

**Proposition 3.13.** *Let  $\lambda$  be a  $2^R$ -parent of  $\mu$  with  $|\mu| < 2^R$ . Then*

$$\eta_\mu^\lambda = N_\lambda(h_\mu^\lambda) - \mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^{R-1}) + \mathbb{I}_{H(\lambda)}(h_\mu^\lambda + 2^{R-1}) + \mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^R - 2^{R-1}).$$

**Remark 3.14.** This proposition tells us that to compute  $\eta_\mu^\lambda$ , we first count the number of entries in  $H(\lambda)$  which lie between  $h_\mu^\lambda$  and  $h_\mu^\lambda - 2^R$ . Then, we add or subtract from this quantity depending on if  $H(\lambda)$  contains  $h_\mu^\lambda + 2^{R-1}$ ,  $h_\mu^\lambda - 2^{R-1}$ , or  $h_\mu^\lambda - 2^R - 2^{R-1}$ . On a number line, if we consider the interval  $[h_\mu^\lambda - 2^R, h_\mu^\lambda]$ , then  $h_\mu^\lambda - 2^{R-1}$  is the midpoint of this interval, and the other two quantities are  $2^R$  away on either side of this midpoint.

*Proof.* For a fixed  $h, x$ , we consider the ratio

$$\rho_h(x) = \frac{\text{Od}(|h - x|)}{\text{Od}(|h - 2^R - x|)}.$$

In this proof, we denote  $h_\mu^\lambda$  simply by  $h$ , and  $x$  is an element of  $H(\lambda)$  not equal to  $h$ . We wish to find all values of  $x$  such that  $\rho_h(x) = -1$ , which allows us to compute  $\eta_\mu^\lambda = \{x \in H(\lambda) : \rho_{h_\mu^\lambda}(x) = -1\}$  by Definition 3.6. We consider three cases: one where  $x$  is smaller than  $h - 2^R$  and  $h$ , the other where  $x$  lies between  $h$  and  $h - 2^R$ , and lastly, where  $x$  is larger than  $h$  and  $h - 2^R$ . In each of these cases, we deal with the absolute value in a different way. The main idea of this proof is to explicitly compute how the 2-valuations are affected, and how the odd part changes.

1. Let  $x < h - 2^R < h$ . The ratio  $\rho_h(x)$  simplifies to  $\frac{\text{Od}(h-x)}{\text{Od}(h-2^R-x)}$ . Let  $h - x = 2^p\omega$  where  $\omega$  is odd and  $p$  is a positive integer strictly smaller than  $R$ . For the numerator of  $\rho_h(x)$ , we have  $\text{Od}(h - x) \equiv \omega \pmod{4}$ . For the denominator, we find  $h - 2^R - x = 2^p\omega - 2^R = 2^p(\omega - 2^{R-p})$ . The part inside the parenthesis is odd as  $2^{R-p}$  is even owing to  $p < R$ . This gives us  $\text{Od}(h - x - 2^R) \equiv \omega - 2^{R-p} \pmod{4}$  and we see  $\rho_h(x) \equiv \omega \cdot (\omega - 2^{R-p}) \pmod{4}$ . If  $p < R - 1$ ,  $2^{R-p}$  is divisible by 4 which makes  $\rho_h(x) \equiv \omega^2 \pmod{4}$ , and thus  $\rho_h(x) = 1$  as the only odd square residue mod 4 is 1. For  $p = R - 1$ ,  $\rho_h(x) \equiv \omega(\omega - 2) \pmod{4}$ , which makes  $\rho_h(x) = -1$ . So, for  $\rho_h(x) = -1$ , we must have  $h - x = 2^{R-1}\omega$ . One one hand, as  $x < h - 2^R$ , we have  $h - x > 2^R$  which shows  $\omega > 2$ . On the other hand  $h$  is bounded by  $n$  (Remark 2.3) which is strictly smaller than  $2^{R+1}$  giving us  $\omega < 4$ . The only possible value of  $\omega$  is 3. So,  $\rho_h(h - 3 \cdot 2^{R-1}) = -1$ , giving us the term  $\mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^R - 2^{R-1})$ .
2. Let  $h - 2^R < h < x$ . This gives  $\rho_h(x) = \frac{\text{Od}(x-h)}{\text{Od}(x-h+2^R)}$ . Assuming  $x - h = 2^p\omega$ , and performing a similar analysis as part 1, we obtain  $\rho_h(x) \equiv \omega(2^{R-p} + \omega) \pmod{4}$ . For  $p < R - 1$ , we again get  $\rho_h(x) = 1$ , while for  $p = R - 1$ , we obtain  $\rho_h(x) = -1$ . The only value of  $\omega$  possible in this case is 1 as  $x$  must be smaller than  $2^{R-1}$  (Remark 2.3). So, we get  $\rho_h(h + 2^{R-1}) = -1$ , and this gives us the term  $\mathbb{I}_{H(\lambda)}(h_\mu^\lambda + 2^{R-1})$ .
3. Let  $h - 2^R < x < h$ . Note that these are exactly the entries enumerated by  $N_\lambda(h_\mu^\lambda)$ . Now,  $\rho_h(x)$  becomes  $\frac{\text{Od}(h-x)}{\text{Od}(-h+2^R+x)}$ . We let  $h - x = 2^p\omega$  as before. In this case, we see

that the denominator is  $\text{Od}(2^R - (h - x)) \equiv 2^{R-p} - \omega \pmod{4}$  while the numerator is simply  $\omega$ . This gives us  $\rho_h(x) \equiv \omega \cdot (2^{R-p} - \omega) \pmod{4}$ . If  $p < R - 1$ , then  $2^{R-p}$  is divisible by 4, rendering  $\rho_h(x) \equiv -\omega^2 \pmod{4}$ . Thus all values of  $x$  where  $p$  is not  $R - 1$  result in  $\rho_h(x) = -1$ . If  $p = R - 1$ , we get  $\rho_h(x) \equiv \omega(2 - \omega) \pmod{4}$  which is always 1. Thus, for exactly  $N_\lambda(h) - \mathbb{I}_{H(\lambda)}(h - 2^{R-1})$  values, we have  $\rho_h(x) = -1$ .

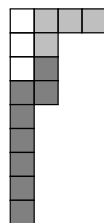
□

**Example 3.15.** Pick  $\mu = (2, 1, 1)$  with  $H(\mu) = \{4, 2, 1\}$  and choose a Type II 8-parent  $\lambda = (2, 2, 2, 2, 1, 1, 1, 1)$  with  $H(\lambda) = \{9, 8, 7, 6, 4, 3, 2, 1\}$  which makes  $h_\mu^\lambda = 8$ . We first compute  $N_\lambda(h_\mu^\lambda)$  which is  $|\{y \in H(\lambda) : 0 < y < 8\}| = 6$ . Also,  $\mathbb{I}_{H(\lambda)}(8 - 4) = 1$  while other indicator functions  $\mathbb{I}_{H(\lambda)}(8 + 4)$  and  $H(\lambda)(8 - 12)$  equal 0. Thus,  $\eta_\mu^\lambda = 6 - 1 = 5$  in this case. Now consider the Type I 16-parent  $\nu = (10, 3, 3, 3, 3, 2, 2, 2)$  of  $\lambda$  such that  $H(\nu) = \{17, 9, 8, 7, 6, 4, 3, 2\}$  and thus  $h_\lambda^\nu = 17$ . Then,  $N_\lambda(h_\lambda^\nu) = |\{y \in H(\nu) : 1 < y < 17\}| = 7$ . As only  $\mathbb{I}_{H(\nu)}(17 - 8) = 1$  we get  $\eta_\lambda^\nu = 7 - 1 = 6$ .

**Remark 3.16.** We now interpret Proposition 3.13 in terms of the hook  $\gamma = \lambda/\mu$  on diagrams. These diagrammatic interpretations will not aid us directly in computations, but offer a visual complement to our methods. Recall from Remark 3.12 that  $N_\lambda(h_\mu^\lambda)$  is one less than the number of rows occupied by  $\gamma$ . Let us denote  $h_\mu^\lambda$  by  $h$ , and as  $h$  is the affected hook-length,  $h - 2^R \notin H(\lambda)$ .

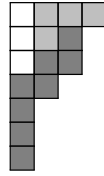
- We first look at  $h + 2^{R-1}$ . The largest entry in the Type I case is  $h$  itself, so  $h + 2^{R-1}$  can only occur in  $H(\lambda)$  in the Type II case, so  $h = 2^{R-1}$ . If  $h + 2^{R-1}$  exists in  $H(\lambda)$  but  $h - 2^R$  does not, then we may remove a hook of size  $2^R + 2^{R-1}$  by sending  $h + 2^{R-1}$  to  $h - 2^R$ . We interpret this as  $\lambda$  being constructed by a two-step process: first, we add a  $2^{R-1}$ -rim hook  $\delta$  to some partition  $\nu$  to obtain  $\mu$ , and then add  $\gamma$  to  $\mu$  to obtain  $\lambda$ . This is done in such a manner that  $\gamma \cup \delta$  is a hook of size  $2^R + 2^{R-1}$  added to  $\nu$ . We can only “extend”  $\gamma$  to  $\gamma \cup \delta$  if the southwesternmost cell of  $\delta$  lies immediately above the northeasternmost cell of  $\gamma$ .

**Example 3.17.** In the following example,  $\nu = (1, 1, 1)$ ,  $\mu = (4, 2, 1)$  is a 4-parent of  $\nu$ , and  $\lambda = (4, 2, 2, 2, 1, 1, 1, 1)$  is an 8-parent of  $\mu$ . We find  $H(\nu) = \{3, 2, 1\}$ ,  $H(\mu) = \{6, 3, 1\}$ , and  $H(\lambda) = \{12, 9, 8, 7, 5, 4, 3, 2, 1\}$ . The following diagram depicts  $\nu$  in white,  $\delta := \mu/\nu$  in light gray and  $\gamma := \lambda/\mu$  in dark gray. Notice how  $\gamma$  can be extended to a hook of size 12 by combining it with  $\delta$ .



As a non-example, we keep the same  $\nu$  and  $\mu$ , but consider  $\hat{\lambda} = (4, 3, 3, 2, 1, 1, 1)$  and notice that  $H(\hat{\lambda}) = \{10, 8, 7, 5, 3, 2, 1\}$  does not contain 12, while also observing that

the dark gray hook cannot be extended to a hook of size 12.



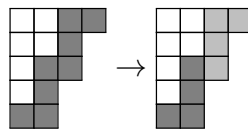
In summary, if our hook  $\gamma$  can be extended to a hook of size  $2^R + 2^{R-1}$ , then  $\mathbb{I}_{H(\lambda)}(h + 2^{R-1}) = 1$ .

- We now look at  $h_\mu^\lambda - 2^{R-1}$  which can only occur in  $H(\lambda)$  when  $2^R \leq h_\mu^\lambda < 2^R + 2^{R-1}$ . One can prove this condition by assuming the negation that if  $h \geq 2^R + 2^{R-1}$  exists in  $H(\lambda)$ , then  $h - 2^{R-1} \geq 2^R$  exists in  $H(\mu)$  where  $|\mu| < 2^R$ , and thus Remark 2.3 is contradicted. If  $h - 2^{R-1}$  is not present in  $H(\lambda)$ , then we may first send  $h$  to  $h - 2^{R-1}$ , and in another step, send  $h - 2^{R-1}$  to  $h - 2^R$ . This corresponds to decomposing the hook  $\gamma$  as a disjoint union of two  $\gamma'$  and  $\gamma''$  where we first remove  $\gamma'$  and then  $\gamma''$ .

**Example 3.18.** Let  $\mu = (2, 2, 1, 1)$  with  $H(\mu) = \{5, 4, 2, 1\}$ . When  $\lambda = (3, 3, 3, 2, 2, 1)$ , we have  $H(\lambda) = \{8, 7, 6, 4, 3, 1\}$  and the aforementioned decomposition of  $\gamma = \lambda/\mu$  (gray) is not possible as  $4 \in H(\lambda)$ .




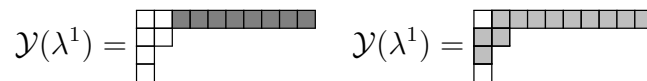
On the other hand, when we have the 8-parent  $\hat{\lambda} = (4, 3, 3, 2, 2)$  with  $H(\hat{\lambda}) = \{8, 6, 5, 3, 2\}$ , we can indeed decompose  $\hat{\gamma} = \hat{\lambda}/\mu$  (gray) into  $\hat{\gamma}'$  (light gray, top) and  $\hat{\gamma}''$  (gray, bottom) as shown



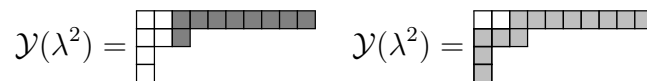
So,  $\mathbb{I}_{H(\lambda)}(h - 2^{R-1}) = 1$  when  $\gamma$  **cannot** be decomposed into two successively removable  $2^{R-1}$ -rim hooks.

- The quantity  $h - 2^R - 2^{R-1}$  can only occur in  $H(\lambda)$  when  $h > 2^R + 2^{R-1}$ , which means  $\lambda$  must be a Type I  $2^R$ -parent, and so we assume this case. If  $h - 2^R - 2^{R-1}$  is not an element of  $H(\lambda)$ , then we may send  $h$  to  $h - 2^R - 2^{R-1}$ , which means  $\gamma$  can be extended (downwards) to a hook  $\gamma' = \gamma \cup \delta$  of size  $2^R + 2^{R-1}$ . As the hook  $\delta$  of size  $2^{R-1}$  is added before  $\gamma$ , the only way for  $\gamma$  to not extend to  $\gamma'$  is that  $\delta$  lies strictly to the south-west of  $\gamma$  in  $\lambda$ . One may check that if this was not the case, then the removal of  $\gamma$  to yield  $\mu$  would contain empty boxes in rows above the cells of  $\delta$ , and thus  $\mu$  would not be a partition shape. We can equivalently state this condition as follows:  $\gamma$  can be extended to a connected subset of  $2^R + 2^{R-1}$  border cells which do not form a hook.

**Example 3.19.** Let  $\mu = (2, 2, 1, 1)$  with  $H(\mu) = \{5, 4, 2, 1\}$  where the diagram of  $\mu$  is constructed after the addition of a 4-rim hook as . Let  $2^R = 8$ , and we consider all four Type I 8-parents of  $\mu$ . For  $\lambda^1 = (10, 2, 1, 1)$ , we have  $H(\lambda^1) = \{13, 4, 2, 1\}$ . In this case, both  $h_\mu^{\lambda^1} = 13$  and  $h_\mu^{\lambda^1} - 2^R - 2^{R-1} = 1$  occur in  $H(\lambda)$ . As we see in the diagram below, the hook  $\gamma$  (in gray) cannot be extended to a removable hook of size 12. We can extend  $\lambda^1/\mu$  to a connected subset of size 12 as shown in light gray, but that is not a hook (as its removal does not preserve partition shape). Here,  $\mathbb{I}_{H(\lambda^1)}(h_\mu^{\lambda^1} - 2^R - 2^{R-1}) = 1$ .



For  $\lambda^2 = (9, 3, 1, 1)$  with  $H(\lambda^2) = \{12, 5, 2, 1\}$ ,  $h_\mu^{\lambda^2} = 12$  exists in  $H(\lambda)$  but  $h_\mu^{\lambda^2} - 12 = 0$  does not. As we see from the diagram,  $\gamma$  can be extended downwards to a hook of size 12, and so  $\mathbb{I}_{H(\lambda^2)}(h_\mu^{\lambda^2} - 2^R - 2^{R-1}) = 0$ .



In the cases of  $\lambda^3 = (7, 3, 3, 1)$  with  $H(\lambda^3) = \{10, 5, 4, 1\}$  and  $\lambda^4 = (6, 3, 3, 2)$  with  $H(\lambda^4) = \{9, 5, 4, 2\}$  we have  $h_\mu^{\lambda^3} = 10$  and  $h_\mu^{\lambda^4} = 9$  respectively but as both these are smaller than 12, the corresponding hooks cannot be extended to border subsets of size 12.



So,  $\mathbb{I}_{H(\lambda)}(h - 2^R - 2^{R-1}) = 1$  if  $\gamma$  can be extended to a non-rim hook border subset of size  $2^R + 2^{R-1}$ , or equivalently, the unique hook of size  $2^{R-1}$  lies strictly south-east of  $\gamma$ .

## 4 Odd-dimensional partitions of Sparse Numbers

In this section, we prove Theorem 1.2 and Corollary 1.3 through heavy use of Proposition 3.9 and 3.13.

### 4.1 The case of sparse numbers

As we have discussed before in Section 1.3, the case of *sparse numbers* is especially amenable to our enumeration methods. In this section, we assume that  $n = 2^R + m$  with  $m < 2^{R-1}$ , which ensures that the binary expansion of  $n$  starts with 10. The immediate effect of this assumption is simplified expressions for the modular hook recursion (Corollary 4.6) and  $\eta_\mu^\lambda$  (Corollary 4.7). Iteratively applying this assumption (for instance, letting  $m = 2^S + l$  with  $l < 2^{S-1}$ ) leads to sparse numbers.

**Definition 4.1.** A natural number  $n$  is called *sparse* if it contains no consecutive 1s in its binary expansion. Equivalently,  $n = [b_k \dots b_0]_2$ , then we have the product  $b_{i+1} \cdot b_i = 0$  for all  $0 \leq i < k$ .

**Example 4.2.** The number  $165 = [10100101]_2$  is sparse while  $91 = [1011011]_2$  is not.

We recall the definitions of  $a(n)$ ,  $a_i(n)$  and  $\delta(n)$  from Section 1.2. For a non-negative integer  $n$ ,  $a(n)$  is the number of odd-dimensional partitions of  $n$ ,  $a_i(n)$  is the number of partitions with dimensions congruent to  $i \pmod 4$ , and  $\delta(n) = a_1(n) - a_3(n)$ . Our proof strategy requires constructing partitions iteratively through the addition of hooks. So, as a base case we look at partitions which are a single hook.

**Definition 4.3.** A partition  $\lambda \vdash n$  is called a *hook partition* if it is of the form  $(n - b, 1, \dots, 1)$  with  $b$  copies of 1 for  $0 \leq b \leq n - 1$ . Equivalently,  $\lambda$  is a hook partition of  $n$  if  $H(\lambda) = \{n, b, b - 1, \dots, 1\}$ .

The dimension of a hook partition  $(n - b, 1, \dots, 1)$  is given by the binomial coefficient  $\binom{n-1}{b}$ , which follows from Proposition 2.9. The modulo 4 properties of binomial coefficients are discussed in the paper [4] by Davis and Webb. The following proposition in their paper is of use to us.

**Proposition 4.4** (Thm. 6 [4]). *Let  $n \in \mathbb{N}$  be **not** sparse, then*

$$\left| \left\{ k \leq n : \binom{n}{k} \equiv 1 \pmod 4 \right\} \right| = \left| \left\{ k \leq n : \binom{n}{k} \equiv 3 \pmod 4 \right\} \right|.$$

**Lemma 4.5.** *We have  $\delta(1) = 1$ ,  $\delta(2) = 2$  and  $\delta(2^R) = 0$  for  $R \geq 2$ .*

*Proof.* The cases  $\delta(1)$  and  $\delta(2)$  can be computed by hand. Now suppose  $R > 1$ . By Remark 2.13, we have that all odd-dimensional partitions of  $2^R$  are hook partitions. Let  $\lambda \vdash 2^R$  such that  $H(\lambda) = \{2^R, b, b - 1, \dots, 1\}$ . By specializing the formula for hook partitions to this case, we get  $f^\lambda = \binom{2^R-1}{b}$ . As  $2^R - 1$  contains only 1s in its binary expansion, it is not a sparse number. We now apply Proposition 4.4 to deduce that  $a_1(2^R) = a_3(2^R)$ , which shows  $\delta(2^R) = 0$  for  $R > 1$ .  $\square$

Let  $n = 2^R + m$  with  $m < 2^{R-1} < 2^R$ . We proceed with the following strategy: choose an odd-dimensional partition  $\mu \vdash m$  as our  $2^R$ -core. Consider all partitions of  $n$  which are the  $2^R$ -parents of  $\mu$ . These are all odd-dimensional partitions by Proposition 2.12. By Proposition 3.9, we can determine  $\delta(n)$  in terms of  $\delta(m)$ . First, we see that Proposition 3.9 simplifies nicely in the  $m < 2^{R-1}$  case.

**Corollary 4.6** (of Proposition 3.9). *Let  $n = 2^R + m$  with  $m < 2^{R-1}$ . For odd-dimensional partitions,  $\lambda \vdash n$  and  $\mu \vdash m$  such that  $\lambda$  is a  $2^R$ -parent of  $\mu$ ,*

$$\text{Od}(f^\lambda) = (-1)^{\eta_\mu^\lambda} \text{Od}(f^\mu).$$

*Proof.* Let  $n$  have the binary expansion  $[b_R b_{R-1} \dots b_0]_2$ . As  $m$  is strictly smaller than  $2^{R-1}$ , it must follow that  $b_{R-1} = 0$ , and the sum of the first two binary digits  $s_2(n) = b_R + b_{R-1}$

must be 1. By Remark 2.3, we know that the maximum element of  $H(\lambda)$  is at most  $n$ , and by the bound on  $m$ , we know that  $n = 2^R + m < 2^R + 2^{R-1}$ . Thus all elements of  $H(\lambda)$  are strictly smaller than  $2^R + 2^{R-1}$ . This gives us the inequality  $2^R \leq h_\mu^\lambda < 2^R + 2^{R-1}$  which establishes  $s_2(h_\mu^\lambda) = 1$ . By plugging the values  $s_2(h_\mu^\lambda) = 1$  and  $s_2(n) = 1$  in  $\text{Od}(f^\lambda) = (-1)^{s_2(n)+s_2(h_\mu^\lambda)+\eta_\mu^\lambda} \text{Od}(f^\mu)$  (Proposition 3.9), we prove the corollary.  $\square$

For  $m < 2^{R-1}$ , even the formula for  $\eta_\mu^\lambda$  in Proposition 3.13 is simplified.

**Corollary 4.7** (of Proposition 3.13). *Let  $n = 2^R + m$  with  $m < 2^{R-1}$ . For odd-dimensional partitions  $\lambda \vdash n$  and  $\mu \vdash m$  such that  $\lambda$  is a  $2^R$ -parent of  $\mu$ ,*

$$\eta_\mu^\lambda = N_\lambda(h_\mu^\lambda) - \mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^{R-1}).$$

*Proof.* As we just established in the proof of Corollary 4.7 above,  $2^R \leq h_\mu^\lambda < 2^R + 2^{R-1}$ . On one hand, as  $h_\mu^\lambda - 2^R - \cdot 2^{R-1}$  is negative, it cannot appear as a hook-length. Thus,  $\mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^R - \cdot 2^{R-1}) = 0$ . On the other hand,  $h_\mu^\lambda$  is at least  $2^R$  which means  $h_\mu^\lambda + 2^{R-1}$  is at least  $2^R + 2^{R-1}$ , which is strictly larger than  $n$ . This shows that  $2^R + 2^{R-1}$  cannot occur as a hook-length of  $H(\lambda)$  by Remark 2.3. This means  $\mathbb{I}_{H(\lambda)}(h_\mu^\lambda + 2^{R-1}) = 0$ . This renders the second and third terms of  $\eta_\mu^\lambda = N_\lambda(h_\mu^\lambda) - \mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^{R-1}) + \mathbb{I}_{H(\lambda)}(h_\mu^\lambda + 2^{R-1}) + \mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^R - 2^{R-1})$  zero, thereby giving us the corollary.  $\square$

**Remark 4.8.** We emphasize that the results of Corollary 4.7 and Corollary 4.6 apply to a subset of numbers strictly larger than sparse numbers. For instance, the above corollaries apply to  $46 = [101110]_2$  even if it is not a sparse number. It is only when we apply the condition of starting with 10 iteratively that sparse numbers emerge. Thus in order to state theorems for a general  $n$ , we also need to know how the recursions simplify when the numbers do not have the binary expansion prefix 10.

## 4.2 Counting $2^R$ -parents

In this section, we fix a partition  $\mu$  and find how many  $2^R$ -parents of  $\mu$  have the same or different dimension modulo 4 compared to  $\mu$ . We start with  $H(\mu)$  and then create  $2^R$ -parents through the hook addition process described in Proposition 2.16. This strategy requires us to consider the case of Type I and Type II parents separately. For Type I parents we only replace elements in  $H(\mu)$  whereas for Type II parents, we replace elements in  $\beta$ -sets arising from the  $r$ -shifts of  $H(\mu)$ . This distinction also indicates that the proof for Type II parents is much longer and more technical than the proof of Type I parents. We establish some notation before we proceed.

**Notation 4.9.** When  $2^R$  is understood, we denote the set of all  $2^R$ -parents of  $\mu$  by  $\mathcal{P}(\mu)$ . Denote the set of Type I and Type II  $2^R$ -parents of  $\mu$  by  $\mathcal{P}_1(\mu)$  and  $\mathcal{P}_2(\mu)$  respectively.

We have  $\mathcal{P}(\mu) = \mathcal{P}_1(\mu) \cup \mathcal{P}_2(\mu)$  and from Proposition 2.16, we know  $|\mathcal{P}_1(\mu)| = \ell(\mu)$  and  $|\mathcal{P}_2(\mu)| = 2^R - \ell(\mu)$ .

**Notation 4.10** (Signed-sum). Let  $\mu$  be a partition and  $\Lambda$  be some set of partitions. Define the *signed-sum*  $S_\Lambda(\mu) = \frac{1}{\text{Od}(f^\mu)} \sum_{\lambda \in \Lambda} \text{Od}(f^\lambda)$ .

The quantity  $S_\Lambda(\mu)$  keeps track of the number of partitions in  $\Lambda$  with the same value of  $\text{Od}$  as  $\mu$  minus the number of partitions in  $\Lambda$  with a different value of  $\text{Od}$ . As will become clearer in the subsequent proofs, accounting for the difference is computationally more useful than enumerating partitions with the same and different  $\text{Od}$  values separately. This choice of definition computes  $\delta(n)$  for  $n = 2^R + m$  where  $m < 2^R$  using the identity  $\delta(n) = \sum_{\lambda \vdash n} \text{Od}(f^\lambda) = \sum_{\mu \vdash m} \text{Od}(f^\mu) S_{\mathcal{P}(\mu)}(\mu)$ . We may sometimes omit the subscript  $\mathcal{P}(\mu)$  and write  $S(\mu)$  for  $S_{\mathcal{P}(\mu)}(\mu)$ . We eventually prove the following theorem.

**Theorem 4.11.** *Let  $n = 2^R + m$  with  $m < 2^{R-1}$ . If  $\mu$  is a partition of  $m$ , then*

$$S_{\mathcal{P}(\mu)}(\mu) = 2 - 2(-1)^m.$$

In order to prove Theorem 4.11, we split the computations for Type I and Type II  $2^R$ -parents. In the case of Type II parents, we further consider two cases. The final proof of the theorem emerges in Section 4.3 after combining the results from Proposition 4.12, and Proposition 4.15.

#### 4.2.1 Type I parents

Recall from Proposition 2.16 that a Type I  $2^R$ -parent of  $\mu$  is constructed by replacing some  $h \in H(\mu)$  by  $h_\mu^\lambda = h + 2^R$ . Furthermore,  $N_\lambda(h_\mu^\lambda)$  counts the number of entries strictly between  $h$  and  $h_\mu^\lambda$ . As we prove in the next proposition, the ratio  $\text{Od}(f^\lambda)/\text{Od}(f^\mu)$  of a Type I  $2^R$ -parent is entirely dictated by  $N_\lambda(h_\mu^\lambda)$  (the number of entries “jumped over”). Using this, we can compute  $S_{\mathcal{P}_1(\mu)}(\mu) = \sum_\lambda \text{Od}(f^\lambda)/\text{Od}(f^\mu)$  where the sum is over all Type I  $2^R$ -parents  $\lambda$  of  $\mu$ .

**Proposition 4.12.** *Let  $\mu$  be an odd-dimensional partition of  $m < 2^{R-1}$ . For any  $\lambda \in \mathcal{P}_1(\mu)$ ,  $h_\mu^\lambda - 2^{R-1} \notin H(\lambda)$  and*

$$S_{\mathcal{P}_1(\mu)}(\mu) = \begin{cases} 0 & \text{if } \ell(\mu) \text{ is even} \\ 1 & \text{if } \ell(\mu) \text{ is odd.} \end{cases}$$

*Proof.* Let  $H(\mu) = \{h_1, \dots, h_k\}$ . By Proposition 2.16, we know that any Type I  $2^R$ -parent  $\lambda$  of  $\mu$  can be constructed by replacing an element  $h \in H(\mu)$  by  $h_\mu^\lambda = h + 2^R$ . Define  $\lambda^{(i)} \in \mathcal{P}_1(\mu)$  such that  $H(\lambda^{(i)}) = H(\mu)[h_i \rightarrow h_i + 2^R]$  for  $1 \leq i \leq k$ .

For any  $i$ , all elements in  $H(\lambda^{(i)})$  are the same as the elements of  $H(\mu)$  except  $h_\mu^{\lambda^{(i)}}$ . Assume  $h_\mu^{\lambda^{(i)}} - 2^{R-1} \in H(\lambda^{(i)})$  then we must have that  $h_\mu^{\lambda^{(i)}} - 2^{R-1} \in H(\mu)$ . As  $h_\mu^{\lambda^{(i)}} > 2^R$ , we have  $h_\mu^\lambda - 2^{R-1} > 2^{R-1} > m$ . By Remark 2.3, this is not possible, and thus  $h_\mu^{\lambda^{(i)}} - 2^{R-1}$  does not occur in  $H(\lambda)$ . This by definition implies  $\mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^{R-1}) = 0$  and the formula in Proposition 3.13 simplifies to  $\eta_\mu^{\lambda^{(i)}} = N_{\lambda^{(i)}}(h_\mu^{\lambda^{(i)}})$ . Using this, we compute

$$S_{\mathcal{P}_1(\mu)}(\mu) = \sum_{\lambda \in \mathcal{P}_1(\mu)} \frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)} = \sum_{\lambda \in \mathcal{P}_1(\mu)} (-1)^{\eta_\mu^\lambda} = \sum_{\lambda \in \mathcal{P}_1(\mu)} (-1)^{N_\lambda(h_\mu^\lambda)}.$$

The second equality is established in Corollary 4.6 while the third equality follows from the immediately preceding discussion.

As all elements of  $H(\mu)$  are strictly smaller than  $m < 2^{R-1}$ , the element  $h_\mu^{\lambda^{(i)}}$  is the largest element in  $H(\lambda^{(i)})$ . So the elements of  $H(\lambda)$  strictly contained between  $h_i$  and  $h_\mu^{\lambda^{(i)}} = h_i + 2^R$  are exactly the elements in  $H(\mu)$  strictly larger than  $h_i$ . These are exactly the elements  $h_1, h_2, \dots, h_{i-1}$ , and thus we compute

$$N_{\lambda^{(i)}}(h_\mu^{\lambda^{(i)}}) = |\{y \in H(\mu) : y > h_i\}| = i - 1.$$

This gives us  $S_{\mathcal{P}_1(\mu)}(\mu) = \sum_{i=1}^k (-1)^{i-1}$ , which by considering the parity of  $\ell(\mu) = \ell(\lambda)$  gives the result. □

**Example 4.13.** We start with the partition  $\mu = (2, 1, 1)$  such that  $H(\mu) = \{4, 2, 1\}$  with  $\text{Od}(f^\mu) = -1$ . Its Type I 8-parents are partitions  $\lambda^{(1)} = (10, 1, 1)$ ,  $\lambda^{(2)} = (8, 3, 1)$  and  $\lambda^{(3)} = (7, 3, 2)$  with  $\beta$ -sets  $\{12, 2, 1\}$ ,  $\{10, 4, 1\}$ , and  $\{9, 4, 2\}$  respectively. To find  $\lambda^{(1)}$ , we replace 4 by 12 in  $H(\mu)$ , and as there are no elements between 4 and 12, we obtain the sign  $(-1)^0 = 1$ . For  $\lambda^{(2)}$ , we replace the 2 by 10 in  $H(\mu)$ , which jumps over the sole element 4, resulting in a sign of  $(-1)^1$ . Computing  $\lambda^{(3)}$  from  $\mu$  requires replacing 1 in  $H(\mu)$  with 9 which jumps over two entries, namely 2 and 4, thus giving us the sign  $(-1)^2$ . We compute  $S_{\mathcal{P}_1(\mu)}(\mu) = 1 + (-1) + 1 = 1$ , which agrees with the proposition as  $\ell(\mu) = 3$  is odd.

**Remark 4.14.** We can interpret Proposition 4.12 in terms of hooks by noticing that the  $i - 1$  in the exponent corresponds to the height of the hook (number of rows occupied minus 1). For each row of the diagram of  $\mu$ , we have a hook. The hook starting in row  $i$  contributes the term  $(-1)^{i-1}$ .

### 4.2.2 Type II parents

The calculation for Type II  $2^R$ -parents is more involved. We eventually prove the following proposition:

**Proposition 4.15.** *Let  $\mu$  be an odd-dimensional partition of  $m < 2^{R-1}$ . Then*

$$S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 2 - 2(-1)^m, & \text{if } \ell(\mu) \text{ is even} \\ 1 - 2(-1)^m, & \text{if } \ell(\mu) \text{ is odd.} \end{cases}$$

Fix  $\mu \vdash m$  and  $n = 2^R + m$  with  $m < 2^{R-1}$  as before. Recall that Type II  $2^R$ -parents are constructed by considering a  $\beta$ -set  $H(\mu)^{+r}$  of  $\mu$  containing 0, and replacing the 0 by  $2^R$ . In terms of the affected hook-length (Definition 3.5), if  $\lambda \in \mathcal{P}_2(\mu)$ , then  $h_\mu^\lambda = 2^R$ . Thus, Corollary 4.7 simplifies to

$$\eta_\mu^\lambda = N_\lambda(2^R) - \mathbb{I}_{H(\lambda)}(2^{R-1}). \tag{4.1}$$

The calculation of this quantity comes with the caveat that not all  $r$ -shifts of  $H(\mu)$  have a corresponding  $2^R$ -parent. For instance, if we want to consider Type II 8-parents of  $\mu = (2, 1, 1)$  with  $H(\mu) = \{4, 2, 1\}$ , we need to find the values of  $r$  for which the  $r$ -shifts do not contain an 8. We see that  $H(\mu)^{+2} = \{6, 4, 3, 1, 0\}$  does not contain an 8, so we can define the corresponding 8-parent  $\{6, 4, 3, 1\}$ . We call this 8-parent  $\lambda_{[2]}$  where the subscript refers to the 2-shift of the  $\beta$ -set. On the other hand,  $H(\mu)^{+4} = \{8, 6, 5, 3, 2, 1, 0\}$  contains an 8 and thus does not have a corresponding 8-parent. This motivates the following definition.

**Notation 4.16.** For  $r \in \{1, \dots, 2^R\}$ , define  $\lambda_{[r]}$  to be the partition for which  $H(\lambda_{[r]})$  can be constructed by replacing 0 by  $2^R$  in  $H(\mu)^{+r}$ . If  $2^R \in H(\mu)^{+r}$ , we define  $\lambda_{[r]} = \mu$ . We will call a value of  $r$  *valid* when  $\lambda_{[r]} \neq \mu$ .

We see from Eq. (4.1), that the existence of  $2^{R-1}$  in  $H(\lambda)$  affects the value of  $\eta_\mu^\lambda$ . We can ensure  $2^{R-1}$  is present in a  $\beta$ -set of  $\mu$  by shifting  $H(\mu)$  by at least  $2^{R-1} + 1$ , as any shift of  $r$  introduces the elements  $0, 1, \dots, r - 1$  in the  $\beta$ -set. We denote the set of these parents by  $\mathcal{P}_2^\downarrow(\mu)$ . On the other hand,  $H(\mu)^{+r}$  when  $1 \leq r \leq 2^{R-1}$  will contain a  $2^{R-1}$  only when there exists an  $h$  in  $H(\mu)$  such that  $h + r = 2^{R-1}$ . For each value of  $h \in H(\mu)$ , we can find  $r_h = 2^{R-1} - h$  such that  $H(\mu)^{+r_h}$  contains  $2^{R-1}$ . So exactly  $\ell(\mu)$  values of  $r$  in  $\{1, 2, \dots, 2^{R-1}\}$  will yield  $2^{R-1} \in H(\mu)^{+r}$  for  $1 \leq r \leq 2^{R-1}$ . This set of  $2^R$ -parents corresponding to the  $r$  values is denoted by  $\mathcal{P}_2^\uparrow(\mu)$ . We describe the two sets in notation, and only keep the parents arising from the valid values of  $r$ .

**Notation 4.17.** Let

$$\mathcal{P}_2^\uparrow(\mu) := \{\lambda_{[r]} : 1 \leq r \leq 2^{R-1}\} \setminus \{\mu\}$$

and

$$\mathcal{P}_2^\downarrow(\mu) := \{\lambda_{[r]} : 2^{R-1} + 1 \leq r \leq 2^R\} \setminus \{\mu\}.$$

We first focus our attention on the  $2^R$ -parents arising from  $r$ -shifts with  $1 \leq r \leq 2^{R-1}$ , that is, the set  $\mathcal{P}_2^\uparrow(\mu)$ . In the proof of the next proposition, we compute  $S_{\mathcal{P}_2^\uparrow(\mu)}(\mu)$ . The sum  $\sum_{\lambda \in \mathcal{P}_2^\uparrow(\mu)} (-1)^{\eta_\mu^\lambda}$  eventually simplifies to an expression dependent on the sum  $\sum_h (-1)^h$  for  $h \in H(\lambda)$  which can be computed by only knowing the parity of the elements of  $H(\mu)$ . Concretely, each even element contributes a  $+1$  while an odd element contributes a  $-1$ . To capture this idea, we define the following notion.

**Definition 4.18** (Parity gap). For a finite  $X \subset \mathbb{N}$ , define the *parity gap*  $\mathcal{G}(X)$  to be the number of even elements of  $X$  minus the number of odd elements of  $X$ , i.e.,  $\mathcal{G}(x) = \sum_{x \in X} (-1)^x$ .

**Example 4.19.** If  $X = \{13, 12, 8, 5, 3, 1, 0\}$ , then  $\mathcal{G}(X) = 3 - 4 = -1$ .

The enumeration result for Type II  $2^R$ -parents of  $\mu$  arising from  $r$ -shifts with  $1 \leq r \leq 2^{R-1}$  takes a concise form dependent on the parity of the elements of  $H(\mu)$ .

**Lemma 4.20.** Let  $\mu$  be an odd-dimensional partition of  $m < 2^{R-1}$ , then

$$S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) = 2(-1)^{\ell(\mu)} \mathcal{G}(H(\mu)).$$

*Proof.* We first show that the largest element in  $H(\lambda_{[r]})$  must be  $2^R$ . The largest element of  $H(\mu)^{+r}$  is  $\max(H(\mu)) + r$  which itself is *strictly* smaller than  $2^R$  as  $\max(H(\mu)) < m < 2^{R-1}$  (Remark 2.3) and  $r \leq 2^{R-1}$ . Thus for  $1 \leq r \leq 2^{R-1}$ , we have  $2^R \notin H(\mu)^{+r}$ , which means that each value of  $r$  in  $\{1, 2, \dots, 2^{R-1}\}$  is valid. So  $|\mathcal{P}_2^\uparrow(\mu)| = 2^{R-1}$ . The first column hook-length set of  $\lambda_{[r]}$ , that is  $H(\lambda_{[r]})$ , is obtained by replacing a 0 in  $H(\mu)^{+r}$  by a  $2^R$ . This makes  $2^R$  the largest element of  $H(\lambda_{[r]})$  which means that  $N_{\lambda_{[r]}}(2^R) = \#\{y \in H(\lambda_{[r]}) : 0 < y < 2^R\}$  counts every element of  $H(\mu)$  except  $2^R$  itself. As an  $r$ -shift introduces  $r$  new elements, including a 0 which is replaced by a  $2^R$ , we see that  $N_{\lambda_{[r]}}(2^R) = \ell(\mu) + r - 1$ . Using Notation 4.10,

we have  $S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) = \sum_{\lambda \in \mathcal{P}_2^\uparrow(\mu)} \frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)}$  which by an application of Corollary 4.6 becomes  $S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) \sum_{\lambda \in \mathcal{P}_2^\uparrow(\mu)} (-1)^{\eta_\mu^\lambda}$ . A direct application of Eq. (4.1) replaces  $\eta_\mu^\lambda$  by  $N_\lambda(2^R) - \mathbb{I}_{H(\lambda)}(2^R)$  to yield

$$\begin{aligned} S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) &= \sum_{\lambda \in \mathcal{P}_2^\uparrow(\mu)} (-1)^{N_\lambda(2^R) - \mathbb{I}_{H(\lambda)}(2^R)} \\ &= \sum_{r=1}^{2^{R-1}} (-1)^{\ell(\mu) + r - 1 - \mathbb{I}_{H(\lambda_{[r]})}(2^{R-1})} \\ &= (-1)^{\ell(\mu) - 1} \sum_{r=1}^{2^{R-1}} (-1)^{r + \mathbb{I}_{H(\lambda_{[r]})}(2^{R-1})}. \end{aligned}$$

The second equality follows from the definition of the set  $\mathcal{P}_2^\uparrow(\mu)$  and our discussion of it. In the third equality, we factor out  $(-1)^{\ell(\mu) - 1}$  and replace  $-\mathbb{I}_{H(\lambda_{[r]})}(2^{R-1})$  with  $+\mathbb{I}_{H(\lambda_{[r]})}(2^{R-1})$  as it preserves the parity of the exponent. The factor  $\mathbb{I}_{H(\lambda_{[r]})}(2^{R-1})$  takes either a value 0 or 1. When it takes the value 1, this corresponds exactly to those  $2^R$ -parents  $\lambda$  in  $\mathcal{P}_2^\uparrow(\mu)$  with  $H(\lambda)$  containing the  $2^{R-1}$ . As  $1 \leq r \leq 2^{R-1}$ , this can only happen if there exists an  $h$  such that  $h+r = 2^{R-1}$ . In fact, for every  $h \in H(\mu)$ , we can choose a unique  $r_h = 2^{R-1} - h$  which ensures that  $2^{R-1} \in H(\mu)^{+r_h}$ . This gives rise to an injective map  $H(\mu) \rightarrow \mathcal{P}_2^\uparrow(\mu)$  given by  $h \mapsto \lambda_{[r_h]}$  where the image contains *all* partitions with  $2^{R-1}$  as a first column hook-length. We now break the sum into two parts depending on whether  $2^{R-1}$  belongs to  $H(\lambda_{[r]})$ , or equivalently whether  $\mathbb{I}_{H(\lambda_{[r]})}(2^{R-1}) = 1$ . Let  $X = \{r_h : h \in H(\mu)\}$  and  $X^c = \{1, \dots, 2^{R-1}\} \setminus X$ .

$$\begin{aligned} S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) &= (-1)^{\ell(\mu) - 1} \left( \sum_{r_h \in X} (-1)^{r_h + \mathbb{I}_{H(\lambda_{[r_h]})}(2^{R-1})} + \sum_{r \in X^c} (-1)^{r + \mathbb{I}_{H(\lambda_{[r]})}(2^{R-1})} \right) \\ &= (-1)^{\ell(\mu) - 1} \left( \sum_{r_h \in X} (-1)^{r_h + 1} + \sum_{r \in X^c} (-1)^r \right) \end{aligned}$$

The second equality follows from the way we defined our set  $X$ , so as to force  $\mathbb{I}_{H(\lambda_{[r_h]})}(2^{R-1})$  to take on the value 1. We add the terms  $\sum_{r \in X} (-1)^r$  to the second sum  $\sum_{r \in X^c} (-1)^r$  to write it as a sum over all values of  $r$ . To ensure we do not change the sum, we also subtract  $\sum_{r_h \in X} (-1)^{r_h}$ . Note that the indexing with the dummy variables  $r$  and  $r_h$  is just for clarity, and does not affect the final outcome.

$$\begin{aligned} S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) &= (-1)^{\ell(\mu) - 1} \left( \sum_{r_h \in X} (-1)^{r_h + 1} - \sum_{r_h \in X} (-1)^{r_h} \right. \\ &\quad \left. + \sum_{r \in X} (-1)^r + \sum_{r \in X^c} (-1)^r \right) \end{aligned}$$

$$= (-1)^{\ell(\mu)-1} \left( \sum_{r_h \in X} ((-1)^{r_h+1} - (-1)^{r_h}) + \sum_{r=1}^{2^{R-1}} (-1)^r \right)$$

As  $R > 1$ , the sum  $\sum_{r=1}^{2^{R-1}} (-1)^r$  is exactly zero. We also observe that  $-(-1)^{r_h} = (-1)^{r_h+1}$ , and thus the terms in the first sum are each counted twice yielding

$$\begin{aligned} S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) &= 2(-1)^{\ell(\mu)-1} \left( \sum_{r_h \in X} (-1)^{r_h+1} + 0 \right) \\ &= 2(-1)^{\ell(\mu)} \left( \sum_{r_h \in X} (-1)^{r_h} \right). \end{aligned}$$

The second equality is obtained by factoring a  $(-1)$  from the terms  $(-1)^{r_h+1}$  inside the sum and multiplying it with the scalar  $(-1)^{\ell(\mu)-1}$  giving  $(-1)^{\ell(\mu)}$ . Now, notice that if we have  $h \in H(\mu)$  and correspondingly an  $r_h$  with  $h+r_h = 2^{R-1}$ , then  $h$  and  $r_h$  have the same parity. Thus we can rewrite the sum over  $r_h \in X$  as a sum over  $h \in H(\mu)$ . By using the definition of the parity gap,  $\mathcal{G}(H(\mu))$ , we finally have

$$S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) = 2(-1)^{\ell(\mu)} \left( \sum_{h \in H(\mu)} (-1)^h \right) = 2(-1)^{\ell(\mu)} \mathcal{G}(H(\mu)). \quad \square$$

The astute reader may wonder why the above statement was relegated to being a lemma rather than a full-fledged proposition. Our work here is not done as we can further analyze the outputs for  $\mathcal{G}(H(\mu))$ . We emphasize the following lemma holds for all odd-dimensional partitions and the sizes are **not** subject to the constraint  $m < 2^{R-1}$ .

**Lemma 4.21.** *For an odd-dimensional partition  $\lambda$ ,*

$$\mathcal{G}(H(\lambda)) = \begin{cases} 0, & \text{if } \ell(\lambda) \text{ is even and } n \text{ is even} \\ 1, & \text{if } \ell(\lambda) \text{ is odd and } n \text{ is even} \\ 2, & \text{if } \ell(\lambda) \text{ is even and } n \text{ is odd} \\ -1, & \text{if } \ell(\lambda) \text{ is odd and } n \text{ is odd} \end{cases}$$

or more concisely,

$$\mathcal{G}(H(\lambda)) = \begin{cases} 1 - (-1)^n, & \text{if } \ell(\lambda) \text{ is even} \\ (-1)^n, & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

*Proof.* Let  $\lambda$  be a  $2^R$ -parent of  $\mu$ . Then by Proposition 2.16, we may find  $s$  satisfying  $0 \leq s \leq 2^R$  and  $h \in H(\mu)^{+s}$  such that  $H(\lambda) \sim_\beta H(\mu)^{+s}[h \rightarrow h + 2^R]$ . In the case of Type I parents, this means replacing some  $h \in H(\mu)$  by  $h + 2^R$ , while in the case of Type II parents, this means  $s$ -shifting the set  $H(\mu)$  and replacing 0 by  $2^R$ . Thus, the only difference between the sets  $H(\lambda)$  and  $H(\mu)^{+s}$  is that  $h$  in  $H(\mu)^{+s}$  is replaced with  $h + 2^R$  in  $H(\lambda)$ .

From the simple fact that  $h$  and  $h + 2^R$  have the same parity, we can deduce that this replacement of elements does not change the overall parity distribution of the set elements and in particular  $\mathcal{G}(H(\lambda)) = \mathcal{G}(H(\mu)^{+s})$ . As  $X^{+1} = \{x + 1 \mid x \in X\} \cup \{0\}$ , we see that under a 1-shift all odd entries become even, all even entries become odd and an additional even entry 0 is introduced. Thus,  $\mathcal{G}(X^{+1}) = 1 - \mathcal{G}(X)$  where the  $-\mathcal{G}(X)$  arises from swapping the parity of entries and the +1 corresponds to the newly introduced zero. Now note that  $\mathcal{G}(X^{+2}) = 1 - \mathcal{G}(X^{+1}) = 1 - (1 - \mathcal{G}(x)) = \mathcal{G}(x)$ . Continuing this, we get

$$\mathcal{G}(X^{+s}) = \begin{cases} \mathcal{G}(X), & \text{if } s \text{ is even} \\ 1 - \mathcal{G}(X), & \text{if } s \text{ is odd.} \end{cases}$$

Assume  $n = 2^{b_l} + \dots + 2^{b_1} + |\alpha|$  with  $b_l > \dots > b_1 > 1$  and  $\alpha = \emptyset$  or  $(1)$ . A  $\beta$ -set of any odd-dimensional partition is constructed as a sequence of “ $s_i$ -shifts followed by incrementing an element by  $2^{b_i}$ ” for  $1 \leq i \leq l$  (Proposition 2.16 and Proposition 2.12). Let  $\lambda$  be constructed successively as

$$\alpha = \nu^0 \subset \nu^1 \subset \dots \nu^l = \lambda$$

where for each  $1 \leq i \leq l$ , we can find  $s_i$  and  $h_i \in H(\nu^{i-1})^{+s_i}$  such that  $H(\nu^i) = H(\nu^{i-1})^{+s_i}[h_i \rightarrow h_i + 2^{b_i}]$ . As all the substitutions of  $h_i$  by  $h_i + 2^{b_i}$  do not affect the parity, we can write

$$\mathcal{G}(H(\lambda)) = \mathcal{G}(X^{+\sum_{i=1}^l s_i})$$

where  $\sum_{i=1}^l s_i$  is number of new entries with respect to  $H(\alpha)$  and is thus equal to  $\ell(\lambda) - \ell(\alpha)$ . So, we have

$$\mathcal{G}(H(\lambda)) = \begin{cases} \mathcal{G}(H(\alpha)), & \text{if } \ell(\lambda) - \ell(\alpha) \text{ is even} \\ 1 - \mathcal{G}(H(\alpha)), & \text{if } \ell(\lambda) - \ell(\alpha) \text{ is odd.} \end{cases}$$

When  $\alpha = \emptyset$ ,  $\mathcal{G}(H(\alpha)) = 0$  and  $\ell(\alpha) = 0$ . In this case,  $n$  is even and we obtain

$$\mathcal{G}(H(\lambda)) = \begin{cases} 0 & \ell(\lambda) \text{ is even} \\ 1 & \ell(\lambda) \text{ is odd.} \end{cases}$$

When  $\alpha = (1)$ , we have  $\mathcal{G}(\{1\}) = -1$  and  $\ell(\alpha) = 1$ . This is the case when  $n$  is odd, so we get

$$\mathcal{G}(H(\lambda)) = \begin{cases} 2, & \text{if } \ell(\lambda) \text{ is even} \\ -1, & \text{if } \ell(\lambda) \text{ is odd.} \end{cases}$$

The concise formula for  $\mathcal{G}(H(\lambda))$  follows from these computations. □

**Example 4.22.** Let  $\lambda, \mu, \pi$  and  $\kappa$  be partitions such that

- $H(\mu) = \{1\}$
- $H(\lambda) = H(\mu)^{+2}[0 \rightarrow 4] = \{4, 3, 1\}$

- $H(\pi) = H(\lambda)^{+6}[0 \rightarrow 8] = \{10, 9, 8, 7, 5, 4, 3, 2, 1\}$
- $H(\kappa) = H(\pi)[5 \rightarrow 21] = \{21, 10, 9, 8, 7, 4, 3, 2, 1\}$ .

The partition  $\kappa$  is a partition of  $16 + 8 + 4 + 1 = 29$  which is an odd number. The length of  $\kappa$  is 9, i.e. odd, as there are 10 entries in  $H(\kappa)$ . By Lemma 4.21, we know that the number of odd numbers must exceed the number of even numbers by 1 in  $H(\kappa)$ . A quick glance at the entries of  $H(\kappa)$  tells us that this is indeed true.

We can now state the result for  $\mathcal{P}_2^\uparrow(\mu)$  in terms of  $m$  with dependence on the length of  $\mu$ .

**Proposition 4.23.** *Let  $\mu$  be an odd-dimensional partition of  $m < 2^{R-1}$   $R \geq 2$ . Then*

$$S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) = \begin{cases} 2(1 + (-1)^{m+1}), & \text{if } \ell(\mu) \text{ is even} \\ 2(-1)^{m+1}, & \text{if } \ell(\mu) \text{ is odd.} \end{cases}$$

*Proof.* From Lemma 4.20, we have the formula  $S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) = 2(-1)^{\ell(\mu)} \mathcal{G}(H(\mu))$ . Now we use the result from Lemma 4.21. When  $\ell(\mu)$  is even,  $(-1)^{\ell(\mu)} = 1$  and  $\mathcal{G}(H(\mu))$  is  $1 - (-1)^m = 1 + (-1)^{m+1}$ . This shows that when  $\ell(\mu)$  is even,  $\mathcal{G}(H(\mu)) = 2(1 + (-1)^{m+1})$  as stated. For the case when  $\ell(\mu)$  is odd, we have  $(-1)^{\ell(\mu)} = -1$  and  $\mathcal{G}(H(\mu)) = (-1)^m$ . This gives us  $\mathcal{G}(H(\mu)) = 2(-1)(-1)^m = 2(-1)^{m+1}$  for partitions  $\mu$  with odd length.  $\square$

With the result for  $r$ -shifts in the domain  $1 \leq r \leq 2^{R-1}$  sorted, we turn our attention towards the values of  $r$  between  $2^{R-1} + 1$  and  $2^R$ . In this case, we see not all values of  $r$  are valid, and thus have no corresponding  $2^R$ -parents (Notation 4.16). In fact, we can show that the values of  $r$  that are **not** valid are exactly  $2^{R-1}$  more than the values  $r_h = 2^{R-1} - h$  as defined in the proof of Lemma 4.20. As we saw in the proof, these  $r_h$ -shift values, and by transitivity the invalid  $r$  values in  $\{2^{R-1}, \dots, 2^R\}$ , are in bijection with the elements of  $H(\mu)$ . The proof of the following proposition is involved but we provide a broad idea through the following hypothetical example. Recall that each  $2^R$ -parent  $\lambda$  has an associated sign given by  $\text{Od}(f^\lambda)/\text{Od}(f^\mu)$ . Considering the example  $R = 4$ , our  $r$ -shifts lie in the region  $9 \leq r \leq 16$ . Recall from Notation 4.16 that  $\lambda_{[r]}$  is the  $2^R$ -parent arising from the  $r$ -shift by replacing 0 in  $H(\mu)^{+r}$  by  $2^R$ . Suppose  $\lambda_{[10]}$  has the sign  $-1$ , and 11, 12, 13 are invalid  $r$  values. If 14 is a valid shift value, then we claim that the sign associated with  $\lambda_{[14]}$  is  $+1$ , that is, the opposite of the immediately preceding  $2^R$ -parent. We show in general that the signs of the  $2^R$ -parents will alternate for consecutive valid shift values (while skipping the invalid values). So as an example if the valid shift values are 9, 10, 14, 16, and the sign of  $\lambda_{[9]}$  is  $+1$ , then the sign of  $\lambda_{[10]}$  is  $-1$ , the sign of  $\lambda_{[14]}$  is  $+1$  and the sign of  $\lambda_{[16]}$  is  $-1$ .

**Proposition 4.24.** *Let  $\mu$  be a partition of  $m < 2^{R-1}$ . For  $2^{R-1} + 1 \leq r \leq 2^R$ , we have  $\lambda_{[r]} \in \mathcal{P}_2^\downarrow(\mu)$  if and only if  $2^{R-1} \notin H(\lambda_{[r-2^{R-1}]})$ . Furthermore,*

$$S_{\mathcal{P}_2^\downarrow(\mu)}(\mu) = \begin{cases} 0, & \text{if } \ell(\mu) \text{ is even} \\ 1, & \text{if } \ell(\mu) \text{ is odd.} \end{cases}$$

*Proof.* For each  $h \in H(\mu)$  we can find an  $r$  such that  $h + r = 2^R$ . These are the only invalid values of  $r$ . Subtracting  $2^{R-1}$  on both sides gives the equality  $h + (r - 2^{R-1}) = 2^{R-1}$  from which the claimed equivalence follows. Now we compute  $S_{\mathcal{P}_2^\downarrow(\mu)}(\mu)$ .

From Corollary 4.7 and Eq. (4.1), we have  $\eta_\mu^\lambda = N_\lambda(2^R) - \mathbb{I}_{H(\lambda)}(2^{R-1})$ . By noticing that a shift of  $r \geq 2^{R-1}$  would force the  $\beta$ -set  $H(\mu)^{+r}$  to always contain a  $2^{R-1}$ , we have  $\eta_\mu^\lambda = N_{\lambda_{[r]}}(2^R) - 1$ . We can now focus on just computing  $N_{\lambda_{[r]}}(2^R)$ , that is, the number of entries in  $H(\lambda_{[r]})$  strictly less than  $2^R$ . Recall from Notation 4.16 that  $\lambda_{[r]}$  is the  $2^R$ -parent arising from the  $r$ -shift by replacing 0 in  $H(\mu)^{+r}$  by  $2^R$ . Suppose  $\lambda_{[r]}, \lambda_{[r+s]} \in \mathcal{P}_2^\downarrow(\mu)$  and  $\lambda_{[r+1]} = \dots = \lambda_{[r+s-1]} = \lambda \notin \mathcal{P}_2^\downarrow(\lambda)$ . In other words,  $r$  and  $r+s$  are valid values of  $r$ -shifts while  $r+1, r+2, \dots, r+s-1$  are not. From our discussion above, a shift  $r+i$  for  $1 \leq i \leq s-1$  can be invalid only if there exists some element  $h \in H(\mu)$  such that  $h+r+i \in H(\mu)^{r+i}$  is equal to  $2^R$ . Call the element  $h_{k+1}$  that satisfies  $h_{k+1} + r + 1 = 2^R$ . Then in order for every successive shift from  $r+1$  to  $r+s-1$  to also be invalid, we must have the consecutive integers  $h_{k+1}-1, h_{k+1}-2, \dots, h_{k+1}-(s-2)$  as elements of  $H(\mu)$ . For conciseness, we define  $h_{k+i} = h_{k+1} - i + 1$  for  $1 \leq i \leq s-1$ . So our first column hook-length set for  $\mu$  takes the form

$$H(\mu) = \{h_1, \dots, h_k, h_{k+1}, \dots, h_{k+s-1}, \dots, h_l\}$$

where  $h_{k+i} + r + i = 2^R$  by definition. Note that we must have  $h_k + r > 2^R$ . We cannot have  $h_k + r = 2^R$  as  $r$  is a valid shift, and if  $h_k + r < 2^R$ , then  $h_{k+1} + r + 1 < h_k + r + 1$  cannot equal  $2^R$ . Similarly, one may argue that  $h_{k+s} + r + s < 2^R$ .

Now in order to compute  $N_{\lambda_{[x]}}$ , the number of entries strictly smaller than  $2^R$  in  $H(\lambda_{[x]})$ , we need to disregard all entries greater than or equal to  $2^R$ . We define a quantity to keep track of these larger entries. Let  $j_x$  be the number of elements strictly bigger than  $2^R$  in  $H(\lambda_{[x]})$ .

When we  $x$ -shift a  $\beta$ -set, we add exactly  $x$  new entries, one of which is zero. If we disregard all entries greater than  $2^R$  as well as the zero (which will be replaced by  $2^R$  in passage to  $\lambda_{[x]}$ ), we get  $N_{\lambda_{[x]}}(2^R) = \ell(\mu) + x - 1 - j_x$ . If we shift by  $r+s$ , we see  $h_{k+i} + r + s > 2^R$  for  $1 \leq i \leq s-1$ , and thus  $j_{r+s} = j_r + s - 1$  as exactly  $s-1$  values exceed  $2^R$  in addition to the values already exceeding  $2^R$  in  $H(\lambda_{[r]})$ . By plugging  $x = r+s$  in the expression for  $N_{\lambda_{[x]}}(2^R)$ , we get  $N_{\lambda_{[r+s]}}(2^R) = \ell(\mu) + (r+s) - 1 - j_{r+s}$  which after using the relationship we just established between  $j_{r+s}$  and  $j_r$ , we get  $N_{\lambda_{[r+s]}}(2^R) = \ell(\mu) + (r+s-1) - (j_r + s - 1)$  which simplifies to  $\ell(\mu) + r - j_r$ . Comparing this to the formula  $N_{\lambda_{[x]}}(2^R)$  for  $x = r$ , we see that  $N_{\lambda_{[r+s]}}(2^R) = N_{\lambda_{[r]}}(2^R) + 1$ . This shows that for each consecutive  $2^R$ -parent of  $\mu$  (when the invalid values are ignored), the value of  $\eta_\mu^\lambda$  increases by 1.

Let  $r_1 < \dots < r_{2^{R-1}-\ell(\mu)}$  be the valid values of  $r$ -shifts, that is,  $\{\lambda_{[r_1]}, \dots, \lambda_{[r_{2^{R-1}-\ell(\mu)}]}\} = \mathcal{P}_2^\downarrow(\mu)$ . By the above discussion, we deduce  $N_{\lambda_{[r_i]}}(2^R) = N_{\lambda_{[r_1]}}(2^R) + i - 1$ . Furthermore, we can express this in terms of  $j_{r_1}$  as

$$N_{\lambda_{[r_i]}}(2^R) = (\ell(\mu) + r_1 - 1 - j_{r_1}) + i - 1 = \ell(\mu) + r_1 + i - j_{r_1} - 2.$$

Suppose  $r_1 = 2^{R-1} + z$  for some  $1 \leq z \leq 2^{R-1}$ , then as  $r_1$  is the smallest valid value, any shift value before this must be invalid. This means that  $j_{r_1} = z - 1$  as each set  $H(\mu)^{+(2^{R-1}+1)}, \dots, H(\mu)^{+(2^{R-1}+z-1)}$  will contain  $2^R$ . This simplifies our expression to give

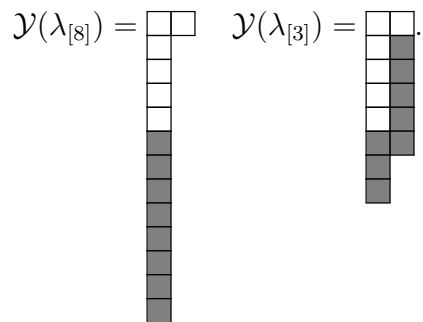
$$N_{\lambda_{[r_i]}}(2^R) = \ell(\mu) + (2^{R-1} + z) + i - (z - 1) - 2 = \ell(\mu) + 2^{R-1} + i - 1.$$

Using the definitions Notation 4.10, Corollary 4.6 and Eq. (4.1), we compute

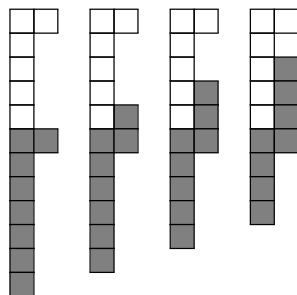
$$\begin{aligned}
 S_{\mathcal{P}_2^\downarrow(\mu)}(\mu) &= \sum_{\lambda \in \mathcal{P}_2^\downarrow(\mu)} (-1)^{\eta_\mu^\lambda} \\
 &= \sum_{i=1}^{2^{R-1}-\ell(\mu)} (-1)^{N_{\lambda_{[r_i]}}(2^R) - \mathbb{I}_{H(\lambda)}(2^{R-1})} \\
 &= \sum_{i=1}^{2^{R-1}-\ell(\mu)} (-1)^{N_{\lambda_{[r_i]}}(2^R) - 1} \\
 &= \sum_{i=1}^{2^{R-1}-\ell(\mu)} (-1)^{(\ell(\mu) + 2^{R-1} + i - 1) - 1}
 \end{aligned}$$

In the exponent of  $(-1)$ , the terms  $2^{R-1} - 2$  do not contribute as they are even owing to  $R \geq 2$ . We factor out  $(-1)^{\ell(\mu)}$  to obtain  $(-1)^{\ell(\mu)} \sum_{i=1}^{2^{R-1}-\ell(\mu)} (-1)^i$ . Looking at the sum,  $\ell(\mu)$  is even, then the terms in the sum pair up and cancel out, while if  $\ell(\mu)$  is odd, we get one positive term  $+1$ . This shows the statement of the proposition.  $\square$

**Remark 4.25.** We can understand the claim in the above proposition that  $\text{Od}(h_\mu^{\lambda_{[r]}})$  and  $\text{Od}(h_\mu^{\lambda_{[r+s]}})$  have opposite signs in terms of hooks on diagrams. Let  $\mu = (2, 1, 1, 1, 1)$  and consider its 8-parents  $\lambda_{[3]}$  and  $\lambda_{[8]}$ . We see that



The hook in  $\lambda_{[8]}$  occupies 8 rows while the hook in  $\lambda_{[3]}$  occupies 7. The following diagrams correspond to the invalid shifts 7, 6, 5, 4 respectively from left to right:



As the shifts decrease by 1 at each step from 7 to 3, the bottommost cell of the hook goes up by one row and so does the topmost cell of the hook, thus maintaining the numbers of rows

occupied as 7. Comparing an 8-shift to a 7-shift, however, we see that the bottom-most cell goes up one row, but the top cell remains in the same row, which decreases the number of rows occupied by 1. This is the diagrammatic reason why the sign flips between successive parents.

The proof of Proposition 4.15 follows from Proposition 4.23 and Proposition 4.24.

*Proof of Proposition 4.15.* From Proposition 4.23,

$$S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) = \begin{cases} 2 - 2(-1)^m, & \text{if } \ell(\mu) \text{ is even} \\ 2(-1)^{m+1}, & \text{if } \ell(\mu) \text{ is odd} \end{cases}$$

and from Proposition 4.24,

$$S_{\mathcal{P}_2^\downarrow(\mu)}(\mu) = \begin{cases} 0, & \text{if } \ell(\mu) \text{ is even} \\ 1, & \text{if } \ell(\mu) \text{ is odd.} \end{cases}$$

As  $S_{\mathcal{P}_2(\mu)}(\mu) = S_{\mathcal{P}_2^\uparrow(\mu)}(\mu) + S_{\mathcal{P}_2^\downarrow(\mu)}(\mu)$ , we add the result of Proposition 4.24 to obtain the result of Proposition 4.15 to show

$$S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 2 - 2(-1)^m, & \text{if } \ell(\mu) \text{ is even} \\ 1 - 2(-1)^m, & \text{if } \ell(\mu) \text{ is odd.} \quad \square \end{cases}$$

### 4.3 Proof of the main theorems

We first establish Theorem 4.11 and then use that result to prove Theorem 1.2.

*Proof of Theorem 4.11.* Recall that  $\mathcal{P}(\mu)$  denotes the set of  $2^R$ -parents of  $\mu$  and  $S_{\mathcal{P}(\mu)}(\mu) = S_{\mathcal{P}_1(\mu)}(\mu) + S_{\mathcal{P}_2(\mu)}(\mu)$ . The result (Proposition 4.12)

$$S_{\mathcal{P}_1(\mu)}(\mu) = \begin{cases} 0 & \text{if } \ell(\mu) \text{ is even} \\ 1 & \text{if } \ell(\mu) \text{ is odd,} \end{cases}$$

when added to the result (Proposition 4.15)

$$S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 2 - 2(-1)^m, & \text{if } \ell(\mu) \text{ is even} \\ 1 - 2(-1)^m, & \text{if } \ell(\mu) \text{ is odd,} \end{cases}$$

gives us  $S_{\mathcal{P}(\mu)}(\mu) = 2 - 2(-1)^m$ . □

We are now ready to prove the main theorem of the paper which establishes the recursion for  $\delta(n)$  when  $n = 2^R + m$  with  $m < 2^{R-1}$ .

*Proof of Theorem 1.2.* To compute  $\delta(n) = a_1(n) - a_3(n)$ , we associate +1 to each partition with dimension 1 modulo 4, and -1 to each partition with dimension 3 modulo 4. This gives

$$\delta(n) = \sum_{\substack{\lambda \vdash n \\ f^\lambda \text{ is odd}}} \text{Od}(f^\lambda).$$

If we collect the partitions  $\lambda$  according to their containment in sets  $\mathcal{P}(\mu)$  for partitions  $\mu \vdash m$ , we get that  $S_{\mathcal{P}(\mu)} = \sum_{\lambda \in \mathcal{P}(\mu)} \frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)}$ , or equivalently  $S_{\mathcal{P}(\mu)} \text{Od}(f^\mu) = \sum_{\lambda \in \mathcal{P}(\mu)} \text{Od}(f^\lambda)$  for each odd-dimensional partition  $\mu$  of  $m$ . As each odd-dimensional  $\lambda \vdash 2^R + m$  appears as a parent of a unique odd-dimensional  $\mu \vdash m$ , we can write

$$\delta(n) = \sum_{\substack{\mu \vdash m \\ f^\mu \text{ is odd}}} S_{\mathcal{P}(\mu)}(\mu) \text{Od}(f^\mu).$$

By using the result  $S_{\mathcal{P}(\mu)}(\mu) = 2 - 2(-1)^m$ , we get

$$\delta(n) = (2 - 2(-1)^m) \sum_{\substack{\mu \vdash m \\ f^\mu \text{ is odd}}} \text{Od}(f^\mu).$$

The sum above, in analogy with  $\delta(n)$ , is equal to  $\delta(m)$ . So, we have

$$\delta(n) = (2 - 2(-1)^m) \delta(m).$$

By the observation that  $n$  and  $m$  have the same parity, we find for  $n \geq 4$

$$\delta(n) = \begin{cases} 0, & \text{if } n \text{ is even} \\ 4\delta(m), & \text{if } n \text{ is odd.} \quad \square \end{cases}$$

*Proof of Corollary 1.3.* Let  $n = 2^{k_1} + 2^{k_2} + \dots + 2^{k_l}$  with  $k_1 > k_2 > \dots > k_l$  be a sparse number; which forces the condition  $k_i > k_{i+1} + 1$  for all  $1 \leq i < l$ . Repeatedly applying Theorem 1.2, we obtain

$$\delta(n) = \delta(n - 2^{k_1}) \cdot \delta(n - 2^{k_1} - 2^{k_2}) \cdot \dots \cdot \delta(2^{k_l}).$$

If  $k_l > 0$  (equivalently,  $n$  is even) then by Lemma 4.5,  $\delta(2^{k_l}) = 0$  and thus  $\delta(n) = 0$ . If  $k_l = 0$  ( $n$  is odd), then  $\delta(2^{k_l}) = \delta(1) = 1$ . Thus, in the odd case, each bit except the rightmost one contributes a factor of 4. The result in the corollary thus follows:

$$\delta(n) = \begin{cases} 2, & \text{if } n = 2 \\ 0, & \text{if } n > 2 \text{ is even} \\ 4^{\nu(n)-1}, & \text{if } n \text{ is odd.} \quad \square \end{cases}$$

## 5 Odd-dimensional partitions of $n = 2^R + 2^{R-1}$

We extend the results of the previous section beyond the sparse case. We consider natural numbers  $n$  of the form  $2^R + 2^{R-1}$  with  $R \geq 1$ . Equivalently, these are numbers  $n$  whose binary expansions are of the form  $[1100\dots]_2$ , i.e.,  $\nu(n) = 2$  and  $D(n) = 1$ . We now prove Theorem 1.4 which states that for  $n = 2^R + 2^{R-1}$  with  $R \geq 1$ ,

$$\delta(n) = \begin{cases} 2, & \text{if } R = 1 \\ 8, & \text{if } R = 2 \\ 0, & \text{otherwise.} \end{cases}$$

We use similar methods as we did in the previous section but with slight modifications. We state a corollary of Proposition 3.9 which is pertinent in this case. Throughout this section, we assume  $R \geq 2$  as Proposition 3.9 requires  $n \geq 4$ . The case of  $R = 1$  is straightforward and can be dealt with by hand.

**Corollary 5.1.** *Let  $n = 2^R + 2^{R-1}$  where  $R \geq 2$ . Let  $\lambda \vdash n$  and  $\mu \vdash 2^{R-1}$  be an odd-dimensional partition such that  $\lambda$  is a  $2^R$ -parent of  $\mu$ . Then*

$$\text{Od}(f^\lambda) = (-1)^{s_2(h_\mu^\lambda) + \eta_\mu^\lambda} \text{Od}(f^\mu).$$

The proof of the above corollary is immediate as the sum of first two digits of  $n$ ,  $s_2(n)$ , is 2. Throughout this section, it follows from Remark 2.3 that  $\mu$  will be a hook partition of  $2^{R-1}$  such that  $H(\mu) = \{2^{R-1}, b, b-1, \dots, 1\}$  for  $0 \leq b \leq 2^{R-1} - 1$ . We first look at Type I  $2^R$ -parents which we denote by  $\mathcal{P}_1(\mu)$  as in Notation 4.9.

**Lemma 5.2.** *Let  $\mu \vdash 2^{R-1}$  be an odd-dimensional partition with  $H(\mu) = \{2^{R-1}, b, b-1, \dots, 1\}$  for  $R \geq 2$  and  $b \geq 0$ . Let  $\mathcal{P}_1(\mu)$  denote the set of  $2^R$ -parents of  $\mu$ , we have*

$$S_{\mathcal{P}_1(\mu)}(\mu) = \begin{cases} 2, & \text{if } \ell(\mu) \text{ is even} \\ 1, & \text{if } \ell(\mu) \text{ is odd.} \end{cases}$$

*Proof.* For each  $x \in H(\mu)$ , define the partition  $\lambda^x$  such that  $H(\lambda^x)$  is obtained by replacing  $x$  by  $x + 2^R$  in  $H(\mu)$ . We start by looking at  $\nu := \lambda^{2^{R-1}}$  for which  $H(\nu)$  is obtained by replacing  $2^{R-1}$  by  $2^R + 2^{R-1}$  in  $H(\mu)$ . This results in  $h_\mu^\nu = 2^R + 2^{R-1}$  and  $s_2(h_\mu^\nu) = 2$ . We compute  $h_\mu^\nu - 2^{R-1} = 2^R$ ,  $h_\mu^\nu - 2^R - 2^{R-1} = 0$ , and  $h_\mu^\nu + 2^{R-1} = 2^{R+1}$ . Plugging these into Proposition 3.13, we have

$$\eta_\mu^\nu = N_\nu(2^R + 2^{R-1}) - \mathbb{I}_{H(\nu)}(2^R) + \mathbb{I}_{H(\nu)}(2^{R+1}) + \mathbb{I}_{H(\nu)}(0).$$

As  $2^{R-1}$  is the largest entry in  $H(\mu)$ , there cannot be any elements between  $2^R + 2^{R-1}$  and  $2^{R-1}$  in  $H(\nu)$ . Thus,

$$N_\nu(2^R + 2^{R-1}) = |\{y \in H(\nu) : 2^{R-1} < y < 2^R + 2^{R-1}\}| = 0.$$

As  $H(\mu)$  does not contain 0,  $2^R$ , or  $2^{R+1}$ ,  $H(\nu) = \{2^R + 2^{R-1}, b, \dots, 1\}$  also does not contain any of these entries. The above form for  $\eta_\mu^\nu$  simplifies to  $N_\nu(2^R + 2^{R-1})$ , which is 0. From this, it follows that  $\text{Od}(f^\nu) = \text{Od}(f^\mu)$ .

Now we consider  $\lambda^x$  for  $1 \leq x \leq b$ . As each  $x$  is strictly smaller than  $2^{R-1}$ , we see that  $x + 2^R < 2^R + 2^{R-1}$  and  $s_2(h_\mu^{\lambda^x}) = 1$ . From Proposition 3.13, it follows that

$$\begin{aligned} \eta_\mu^{\lambda^x} &= N_{\lambda^x}(2^R + x) - \mathbb{I}_{H(\lambda^x)}(x + 2^{R-1}) + \mathbb{I}_{H(\lambda^x)}(x + 2^R + 2^{R-1}) \\ &\quad + \mathbb{I}_{H(\lambda^x)}(x - 2^{R-1}). \end{aligned}$$

Explicitly,  $H(\lambda^x) = \{x + 2^R, 2^{R-1}, b, \dots, x + 1, x - 1, \dots, 1\}$ . There are  $b + 1 - x$  entries in  $H(\lambda^x)$  that lie strictly between  $x + 2^R$  and  $x$ ; namely  $x + 1, x + 2, \dots, b$  and  $2^{R-1}$ . From this, we find  $N_{\lambda^x}(2^R + x) = b + 1 - x$ . By looking at the elements of  $H(\mu)$ , we can conclude that  $H(\lambda^x)$  does not contain  $x - 2^{R-1}$ ,  $x + 2^{R-1}$  or  $x + 2^R + 2^{R-1}$ . Thus,  $\eta_\mu^{\lambda^x} = N_\mu^{\lambda^x} = b + 1 - x$ .

For all elements  $x \in H(\mu)$ , we have  $\text{Od}(f^{\lambda^x})/\text{Od}(f^\mu) = (-1)^{\eta_\mu^{\lambda^x}} + 1$  where  $\eta_\mu^{\lambda^x} = b - x + 1$  when  $x < 2^{R-1}$  and 0 when  $x = 2^{R-1}$ , and the +1 is because  $s_2(h_\mu^{\lambda^x}) = 1$ .

Now we get (as in Notation 4.10),

$$S_{\mathcal{P}_1(\mu)}(\mu) = 1 + \sum_{x=1}^b (-1)^{b+1-x+1} = 1 + \sum_{x=1}^b (-1)^{b-x} = 1 + \sum_{y=0}^{b-1} (-1)^y$$

where the last equality follows by setting  $y = b - x$  in the sum. As we have elements  $1, 2, \dots, b$  in  $H(\mu)$ , the value  $b$  is one less than  $\ell(\mu)$  and has the opposite parity to  $\ell(\mu)$ . This means that the upper bound  $b - 1$  has the same parity as that of  $\ell(\mu)$ . Thus, if the parity of  $\ell(\mu)$  is even,  $S_{\mathcal{P}_1(\mu)}(\mu) = 1 + 1 = 2$  and if the parity of  $\ell(\mu)$  is odd,  $S_{\mathcal{P}_1(\mu)}(\mu) = 1 + 0 = 1$ .  $\square$

**Remark 5.3.** We may interpret the above result in terms of hooks according to the conditions in Remark 3.16. We notice that  $\mathbb{I}_{H(\lambda^x)}(h_\mu^{\lambda^x} - 2^R - 2^{R-1}) = 0$  as hooks cannot be extended to non-rim hook subsets of size  $2^R + 2^{R-1}$ . This obstruction is because it requires all cells of  $\lambda^x$  to be on the border which itself should not form a hook shape, but that is not possible. Also, as we are considering Type I parents, we find  $\mathbb{I}_{H(\lambda^x)}(h_\mu^{\lambda^x} + 2^{R-1}) = 0$ , as the hooks cannot be extended upwards. Now, we notice that all hooks  $\lambda^x/\mu$  can be decomposed into two  $2^{R-1}$ -rim hooks as the top row of  $\lambda^x/\mu$  always contains at least  $2^{R-1} + 1$  boxes, which can be shown by size considerations. This makes  $\mathbb{I}_{H(\lambda^x)}(h_\mu^{\lambda^x} - 2^{R-1}) = 0$ . In the following example,  $\mu = (2, 1, 1)$ :

$$\mathcal{Y}(\lambda^1) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \quad \mathcal{Y}(\lambda^2) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array} \quad \mathcal{Y}(\lambda^3) = \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline \end{array}$$

We now compute the signed-sum for Type II  $2^R$ -parents of  $\mu$ . The  $r$ -shifts of concern are the ones which lead to the existence of  $2^{R-1}$ ,  $2^R$  or  $2^R + 2^{R-1}$  in  $H(\mu)^{+r}$ . We see in the proof that these conditions lead to splitting of  $1 \leq r \leq 2^R$  into six disjoint intervals.

**Lemma 5.4.** *Let  $\mu \vdash 2^{R-1}$  be an odd-dimensional partition with  $H(\mu) = \{2^{R-1}, b, b - 1, \dots, 1\}$  for  $R \geq 2$  and  $b \geq 0$ . Let  $\mathcal{P}_2(\mu)$  denote the set of  $2^R$ -parents of  $\mu$ , then*

$$S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 2, & \text{if } \ell(\mu) \text{ is even} \\ 3, & \text{if } \ell(\mu) \text{ is odd.} \end{cases}$$

*Proof.* Denote by  $\lambda_{[r]}$  the partition for which  $H(\lambda_{[r]})$  is obtained by replacing 0 with  $2^R$  in  $H(\mu)^{+r}$  when  $2^R \notin H(\mu)^{+r}$ . These are exactly the elements of  $\mathcal{P}_2(\mu)$ . For an invalid shift  $s$  (Notation 4.16), we let  $\lambda_{[s]}$  be  $\mu$ . First, for all  $\lambda \in \mathcal{P}_2(\mu)$ , the affected hook-length (Definition 3.5),  $h_\mu^\lambda$ , is  $2^R$ , and so  $s_2(h_\mu^\lambda) = 1$ . This reduces the computation of the signed sum as follows:

$$S_{\mathcal{P}_2(\mu)}(\mu) = \sum_{\lambda \in \mathcal{P}_2(\mu)} (-1)^{\eta_\mu^{\lambda}+1} = (-1) \cdot \sum_{\lambda \in \mathcal{P}_2(\mu)} (-1)^{\eta_\mu^{\lambda}}$$

As  $2^R - (2^R + 2^{R-1}) < 0$ , Proposition 3.13 tells us that  $\eta_\mu^\lambda = N_\lambda(2^R) - \mathbb{I}_{H(\lambda)}(2^{R-1}) + \mathbb{I}_{H(\lambda)}(2^R + 2^{R-1})$  for any  $\lambda \in \mathcal{P}_2(\mu)$ .

In order to deduce whether  $2^{R-1}$ ,  $2^R$  or  $2^R + 2^{R-1}$ , we divide the set  $\{1, 2, \dots, 2^R\}$  of values that  $r$  can take into six intervals, of which some may be empty. As one may check, the following cases work for all values of  $b \geq 0$ .

1. Consider the case when  $1 \leq r \leq 2^{R-1} - b - 1$ . The largest entry in  $H(\mu)^{+r}$  is  $2^{R-1} + r$  which is at most  $2^R - b - 1$  which itself is strictly smaller than  $2^R$ . The second largest element in  $H(\mu)^{+r}$  must be  $b + r$  which is strictly smaller than  $2^{R-1}$ . Therefore,  $H(\lambda_{[r]})$  has the maximum element as  $2^R$  and does not contain  $2^{R-1}$ . This forces  $\mathbb{I}_{H(\lambda)}(2^{R-1})$  and  $\mathbb{I}_{H(\lambda)}(2^R + 2^{R-1})$  to be zero. So, we have  $\eta_\mu^{\lambda_{[r]}} = N_{\lambda_{[r]}}(2^R)$  which can be computed by the number of entries smaller than  $2^R$ . This is exactly one less than the number of entries in  $H(\lambda)$ . The length of  $\lambda$  is  $r$  more than the length of  $\mu$  which gives us  $\eta_\mu^{\lambda_{[r]}} = \ell(\mu) + r - 1$ .
2. Consider  $2^{R-1} - b \leq r \leq 2^{R-1} - 1$ . If  $b = 0$ , then the interval in this case is empty. If  $b$  is positive, then  $H(\mu)^{+r}$  contains  $2^{R-1}$  which gives  $\mathbb{I}_{H(\lambda)}(2^{R-1}) = 1$ . On the other hand, as the largest value in  $H(\mu)^{+r}$  is  $b + r$ , which is strictly smaller than  $2^R + 2^{R-1}$ ,  $\mathbb{I}_{H(\lambda)}(2^R + 2^{R-1}) = 0$ . So  $\eta_\mu^{\lambda_{[r]}} = N_{\lambda_{[r]}}(2^R) - 1$ . We can compute  $\ell(\mu) + r - 1$  similarly to part 1 and subtract the contribution from  $\mathbb{I}_{H(\lambda)}(2^{R-1})$  to get  $\eta_\mu^{\lambda_{[r]}} = \ell(\mu) + r - 2$ .
3. When  $r = 2^{R-1}$ , we get  $2^{R-1} + 2^{R-1} = 2^R \in H(\mu)^{+2^{R-1}}$ , so  $\lambda_{[2^{R-1}]} \notin \mathcal{P}_2(\mu)$ .
4. In the case when  $2^{R-1} + 1 \leq r \leq 2^R - b - 1$ ,  $H(\lambda_{[r]})$  always contains  $2^{R-1}$ , so  $\mathbb{I}_{H(\lambda)}(2^{R-1}) = 1$ . Furthermore, the largest element of  $H(\lambda_{[r]})$  is  $2^{R-1} + r \leq 2^{R-1} + 2^R - b - 1$  which is smaller than  $2^R + 2^{R-1}$ , and thus  $\mathbb{I}_{H(\lambda)}(2^R + 2^{R-1}) = 0$ . The number of elements in  $H(\lambda_{[r]})$  are  $\ell(\mu) + r$ , out of which one element is  $2^R$  and the other element is  $2^{R-1} + 2^R - b - 1$ , which is greater than  $2^R$ . This shows that the number of entries in  $H(\lambda_{[r]})$  strictly smaller than  $2^R$ , that is  $N_{\lambda_{[r]}}(2^R)$ , is  $\ell(\mu) + r - 2$ . From this, we obtain  $\eta_\mu^{\lambda_{[r]}} = (\ell(\mu) + r - 2) - 1 = \ell(\mu) + r - 3$ .
5. For all  $2^R - b \leq r \leq 2^R - 1$ , we have  $2^R \in H(\mu)^{+r}$ , so  $\lambda_{[r]} \notin \mathcal{P}_2(\mu)$ .
6. Finally, when  $r = 2^R$ , the set  $H(\mu)^{+2^R}$  contains both  $2^{R-1}$  and  $2^R + 2^{R-1}$ . Furthermore, the set contains  $1, 2, \dots, 2^R - 1$  as elements which means  $N_{\lambda_{[2^R]}}(2^R) = 2^R - 1$ . So,  $\eta_\mu^{\lambda_{[2^R]}} = N_{\lambda_{[2^R]}}(2^R) = (2^R - 1) - 1 + 1 = 2^R - 1$ .

We sum over the intervals in (1), (2), (4) and (6) to give

$$S_{\mathcal{P}_2(\mu)}(\mu) = (-1) \cdot \left( \sum_{r=1}^{2^{R-1}-b-1} (-1)^{\ell(\mu)+r-1} + \sum_{r=2^{R-1}-b}^{2^{R-1}-1} (-1)^{\ell(\mu)+r-2} + \sum_{r=2^{R-1}+1}^{2^R-b-1} (-1)^{\ell(\mu)+r-3} + (-1)^{2^R-1} \right).$$

We multiply the  $-1$  inside the sum to find

$$S_{\mathcal{P}_2(\mu)}(\mu) = \sum_{r=1}^{2^{R-1}-b-1} (-1)^{\ell(\mu)+r} + \sum_{r=2^{R-1}-b}^{2^{R-1}-1} (-1)^{\ell(\mu)+r-1}$$

$$+ \sum_{r=2^{R-1}+1}^{2^R-b-1} (-1)^{\ell(\mu)+r-2} + (-1)^{2^R}.$$

The last term is +1, which we keep to the side, and from the rest of the terms, we can factor out  $(-1)^{\ell(\mu)}$ , to get

$$S_{\mathcal{P}_2(\mu)}(\mu) = (-1)^{\ell(\mu)} \left( \sum_{r=1}^{2^{R-1}-b-1} (-1)^r + \sum_{r=2^{R-1}-b}^{2^{R-1}-1} (-1)^{r-1} + \sum_{r=2^{R-1}+1}^{2^R-b-1} (-1)^{r-2} \right) + 1.$$

The exponent  $r - 2$  in the third sum can be replaced with  $r$  while keeping the same bounds as the parity is preserved. This simple change gives us

$$S_{\mathcal{P}_2(\mu)}(\mu) = (-1)^{\ell(\mu)} \left( \sum_{r=1}^{2^{R-1}-b-1} (-1)^r + \sum_{r=2^{R-1}-b}^{2^{R-1}-1} (-1)^{r-1} + \sum_{r=2^{R-1}+1}^{2^R-b-1} (-1)^r \right) + 1,$$

and we notice that the third sum is the same as the first sum but the bounds are shifted up by  $2^{R-1}$ . Adding  $2^{R-1}$  in the exponent does not change the sign, so the two sums are equal. We now have

$$S_{\mathcal{P}_2(\mu)}(\mu) = 1 + 2(-1)^{\ell(\mu)} \sum_{r=1}^{2^{R-1}-b-1} (-1)^r + \sum_{r=2^{R-1}-b}^{2^{R-1}-1} (-1)^{r-1}.$$

The  $2^{R-1}$  in the bounds of the sum on the right hand side does not effect the value of the sum as  $R \geq 2$ . We reconsider the sum as ranging from  $-b$  to  $-1$ , and the exponent of  $-1$  ranges from  $-(b+1)$  to  $-2$ . As  $-1$  is its own inverse, we may write the sum as ranging over  $x' = 2$  to  $b+1$ . A shift of 2 preserves the parity, and thus we may rewrite the sum as ranging over  $x = 0$  to  $x = b-1$ . We have

$$S_{\mathcal{P}_2(\mu)}(\mu) = 1 + 2(-1)^{\ell(\mu)} \sum_{r=1}^{2^{R-1}-b-1} (-1)^r + \sum_{x=0}^{b-1} (-1)^x.$$

The number of elements in  $H(\mu)$  is one more than  $b$  by definition, and so  $b = \ell(\mu) - 1$ . This allows us to rewrite our sums in terms of  $\ell(\mu)$  as

$$S_{\mathcal{P}_2(\mu)}(\mu) = 1 + 2(-1)^{\ell(\mu)} \sum_{r=1}^{2^{R-1}-\ell(\mu)} (-1)^r + \sum_{x=0}^{\ell(\mu)-2} (-1)^x.$$

When  $\ell(\mu)$  is even, we get  $S_{\mathcal{P}_2(\mu)}(\mu) = 1 + 2 \cdot 1 \cdot 0 + 1 = 2$ . When  $\ell(\mu)$  is odd, we get  $S_{\mathcal{P}_2(\mu)}(\mu) = 1 + 2 \cdot (-1) \cdot (-1) + 0 = 3$ . □

**Remark 5.5.** We interpret the intervals in terms of diagrams for  $\gamma = \lambda/\mu$ . The interval (1) corresponds to the case where  $\gamma$  contains more than  $2^{R-1}$  boxes in the top row of  $\lambda$  and (2) corresponds to the case where  $\gamma$  contains at most  $2^{R-1}$ , but still positive, number of boxes in the top row of  $\lambda$ . The interval (4) corresponds to  $\gamma$  containing boxes weakly below the second row (if it exists) of  $\lambda$ , and (6) is when all boxes of  $\gamma$  are contained in a column below the bottom row of  $\lambda$ .

With all of these ingredients, we are ready to prove Theorem 1.4.

*Proof of Theorem 1.4.* Combining the results

$$S_{\mathcal{P}_1(\mu)}(\mu) = \begin{cases} 2, & \text{if } \ell(\mu) \text{ is even} \\ 1, & \text{if } \ell(\mu) \text{ is odd} \end{cases}$$

and

$$S_{\mathcal{P}_2(\mu)}(\mu) = \begin{cases} 2, & \text{if } \ell(\mu) \text{ is even} \\ 3, & \text{if } \ell(\mu) \text{ is odd} \end{cases}$$

from Lemma 5.2 and Lemma 5.4 respectively, we see that  $S_{\mathcal{P}(\mu)}(\mu) = 4$  when  $\mu \vdash 2^{R-1}$ . Thus, for  $n = 2^R + 2^{R-1}$ , we find

$$\begin{aligned} \delta(n) &= \sum_{\substack{\lambda \vdash n \\ \lambda \text{ is odd}}} \text{Od}(f^\lambda) = \sum_{\substack{\mu \vdash 2^{R-1} \\ \mu \text{ is odd}}} S_{\mathcal{P}(\mu)}(\mu) \text{Od}(f^\mu) \\ &= 4 \sum_{\substack{\mu \vdash 2^{R-1} \\ \mu \text{ is odd}}} \text{Od}(f^\mu) = 4\delta(2^{R-1}). \end{aligned}$$

From Lemma 4.5, we know that  $\delta(2^{R-1}) = 0$  for  $R \geq 2$ . This proves our theorem except when  $n = 3$ , which can be shown by explicitly computing  $\delta(3)$  for the partitions  $(1, 1, 1)$ ,  $(2, 1)$  and  $(3)$ . Out of these  $(1, 1, 1)$  and  $(3)$  are odd-dimensional, and have dimensions equal to 1, thus, yielding  $\delta(n) = 2$ .  $\square$

## 6 Tackling the case for general $n$

For this section, we suppose that  $n = 2^R + m$  with  $2^{R-1} \leq m < 2^R$ . If we could find a recursive expression for  $\delta(n)$  for such an  $n$ , then we can compute the values of  $\delta(k)$  for any natural number  $k$  by using the results established in the previous sections. This question, however, is open. In this case,  $s_2(n) = 2$ , and so for any  $2^R$ -parent  $\lambda$  of an odd-dimensional partition  $\mu$  of  $m$ ,

$$\frac{\text{Od}(f^\lambda)}{\text{Od}(f^\mu)} = (-1)^{s_2(h_\mu^\lambda) + \eta_\mu^\lambda}.$$

In the previous cases, we saw that  $S_{\mathcal{P}(\mu)}(\mu)$  was only dependent on the size of  $\mu$  and not on the specific partition chosen. In the unsolved case for the general  $n$ , this does not necessarily hold true. Consider the following table for odd-dimensional partitions of 6 with  $2^R = 8$ . Each odd-dimensional partition of 6 has dimension 1 modulo 4, and so we find  $\delta(14) = 0$ .

$\mu$	$H(\mu)$	$S_{\mathcal{P}(\mu)}(\mu)$
(6)	{6}	4
(5,1)	{6,1}	-4
(4,2)	{5,2}	-4
(3,3)	{4,3}	4
(2,2,2)	{4,3,2}	4
(2,2,1,1)	{5,4,2,1}	-4
(2,1,1,1,1)	{6,4,3,2,1}	-4
(1,1,1,1,1,1)	{6,5,4,3,2,1}	4.

**Remark 6.1.** It is not the case that  $S_{\mathcal{P}(\mu)}(\mu)$  is always either 4, -4 or 0. When  $\mu = (31, 15, 4, 3, 1) \vdash 54$  with  $2^R = 64$ , we have  $S_{\mathcal{P}(\mu)}(\mu) = 12$ .

The full computation of  $S(\mu)$  for a partition  $\mu$  is not illuminating but we aim to show where the difficulty in the general case lies. Consider  $\mu = (2, 2, 1, 1)$  with  $H(\mu) = \{5, 4, 2, 1\}$ . Consider its Type I 8-parent  $\lambda$  with  $H(\lambda) = \{9, 5, 4, 2\}$  such that  $h_\mu^\lambda = 9$ . In this case,  $s_2(h_\mu^\lambda) = 1$ . As  $h_\mu^\lambda$  is the largest entry in  $H(\lambda)$ , we have

$$\eta_\mu^\lambda = N_\lambda(h_\mu^\lambda) - \mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^{R-1}) + \mathbb{I}_{H(\lambda)}(h_\mu^\lambda - 2^R - 2^{R-1}) = N_\lambda(9) - \mathbb{I}_{H(\lambda)}(5) + \mathbb{I}_{H(\lambda)}(-3).$$

The appearance of the term  $\mathbb{I}_{H(\lambda)}(5)$  could have been easily deduced from  $H(\mu)$  by noticing that 1 exists in  $H(\mu)$  and  $5 - 1 = 4$ . Thus, when 1 is replaced by  $1 + 8 = 9$ , we will need to check if  $9 - 4 = 5$  exists in  $H(\mu)$ . This follows from the simple fact that  $h_\mu^\lambda - 2^{R-1}$  exists in  $H(\lambda)$  if and only if  $h_\mu^\lambda - 2^{R-1}$  exists in  $H(\mu)$ . In particular, the existence of  $h_\mu^\lambda - 2^R - 2^{R-1}$  in  $H(\lambda)$  is tied to  $h := h_\mu^\lambda - 2^{R-1}$  and  $h - 2^{R-1}$  both existing in  $H(\mu)$ .

Let us look at another 8-parent  $\rho$  with  $H(\rho) = \{13, 4, 2, 1\}$ . In this case,  $h_\mu^\lambda = 13$  and while  $h_\mu^\lambda - 2^{R-1} = 9$  does not exist in  $H(\lambda)$  but  $h_\mu^\lambda - 2^R - 2^{R-1} = 1$  does. We see that this again stems from the fact that both 5 and 1 exist in  $H(\mu)$ .

This presents a possible strategy to attack the general case. We categorize the elements of  $H(\mu) = \{h'_1, \dots, h'_s, h_1, \dots, h_u\}$  into two parts such that  $h'_i < 2^{R-1}$  and  $h_j \geq 2^{R-1}$ . Under this segregation, we can pair up elements which differ by  $2^{R-1}$ , and it is by recording the contributions of these internally interacting pairs we can handle the general case. In the above example for  $H(\mu)$ ,  $5, 4 \geq 2^{R-1}$  while  $2, 1 < 2^{R-1}$ . The paired up elements in this case are 1 and 5, which differ by 4. In the case handled in Section 4 all entries of the  $\beta$ -set were strictly smaller than  $2^{R-1}$  and thus  $s = 0$  and  $u = \ell(\mu)$ . On the other hand, for the case handled in Section 5, we had exactly one entry equal to  $2^{R-1}$  while the rest were smaller, so in our above notation  $s = 1$  while  $u = \ell(\mu) - 1$ . In both these cases, there are no entries which are separated by  $2^{R-1}$ , which made our analysis simpler.

We end this section with one important observation. Assume  $n = 2^R + m$  with  $2^{R-1} \leq m < 2^R$ . Let  $\mu$  be an odd-dimensional partition of  $m$ , then  $\mu$  contains a unique  $2^{R-1}$ -rim hook, which means that there exists some  $1 \leq t \leq s$  such that  $h'_t - 2^{R-1}$  is not in  $H(\mu)$ . This implies that for all other  $h'_i$  with  $i \neq t$ , there exists some  $1 \leq v_i \leq u$  such that  $h'_i - 2^{R-1} = h_{v_i}$ . So, we can remove a  $2^{R-1}$ -rim hook from  $h'_t$  but not from other  $h'_i$ . In the case covered in Section 5, we only had  $t = u = 1$ , which made it easier to analyze. We claim that it is due to the varying distribution of these internally interacting pairs  $(h'_i, h_{v_i})$  for  $1 \leq i \neq t \leq s$ , that the values of  $S_{\mathcal{P}(\mu)}$  vary for  $\mu$ .

## 7 Data for $\delta(n)$

We saw that  $\delta(n)$  was a power of 4 for sparse numbers but this does not hold in general as can be seen in the table below. In general,  $\delta(n)$  may not be even be a power of 2 as  $\delta(118) = -384$ . The run from 124 to 127 shows how egregious this deviation can get. We remark that  $\delta(n) = a_1(n) - a_3(n)$  is even by noticing that for  $n \geq 2$ ,  $a(n) = a_1(n) + a_3(n)$  is even by McKay’s result, and thus  $a_1(n)$  and  $a_3(n)$  have the same parity.

$n$	$\delta(n)$	$n$	$\delta(n)$	$n$	$\delta(n)$	$n$	$\delta(n)$
0	1	32	0	64	0	96	0
1	1	33	4	65	4	97	0
2	2	34	0	66	0	98	0
3	2	35	8	67	8	99	0
4	0	36	0	68	0	100	0
5	4	37	16	69	16	101	0
6	8	38	0	70	0	102	0
7	0	39	0	71	0	103	0
8	0	40	0	72	0	104	0
9	4	41	16	73	16	105	0
10	0	42	0	74	0	106	0
11	8	43	32	75	32	107	0
12	0	44	0	76	0	108	0
13	0	45	0	77	0	109	0
14	0	46	0	78	0	110	0
15	0	47	0	79	0	111	0
16	0	48	0	80	0	112	0
17	4	49	0	81	16	113	32
18	0	50	0	82	0	114	-64
19	8	51	0	83	32	115	64
20	0	52	0	84	0	116	-64
21	16	53	0	85	64	117	64
22	0	54	0	86	0	118	-384
23	0	55	0	87	0	119	128
24	0	56	0	88	0	120	256
25	0	57	32	89	0	121	64
26	0	58	-64	90	0	122	-256
27	0	59	64	91	0	123	-256
28	0	60	64	92	0	124	768
29	32	61	-128	93	128	125	640
30	-64	62	-256	94	0	126	168
31	0	63	128	95	0	127	256

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## References

- [1] Amrutha, P. & Geetha, T., “On the degrees of representations of groups not divisible by  $2^k$ ”, *Journal Of Algebra And Its Applications* **23** (2024).
- [2] Ayyer, A., Prasad, A. & Spallone, S., “Odd partitions in Young’s lattice”, *arXiv* (2016).
- [3] Cabanes, M. & Späth, B., “The McKay Conjecture on character degrees”, *arXiv* (2025).
- [4] Davis, K. & Webb, W., “Pascal’s Triangle modulo 4”, *Fibonacci Quarterly* **29** (1989), pp. 79–83.
- [5] Egecioglu, Ö. & Remmel, J., “A combinatorial interpretation of the inverse kostka matrix”, *Linear And Multilinear Algebra* **26** (1990), pp. 59–84.
- [6] Frame, J., Robinson, G. & Thrall, R., “The Hook Graphs of the Symmetric Group”, *Canadian Journal Of Mathematics* **6** (1954), pp. 316–324.
- [7] Frobenius, G. “Über die Charaktere der symmetrischen Gruppe”, *Königliche Akademie der Wissenschaften* (1900).
- [8] Ganguly, J. & Spallone, S., “Spinorial representations of symmetric groups”, *Journal Of Algebra* **544** (2020), pp. 29–46.
- [9] Gordon, J., Kerber, A., *The Representation Theory of the Symmetric Group*, Cambridge University Press (1984).
- [10] Huang, J., “Dimensions of Compositions Modulo a Prime”, *The Electronic Journal Of Combinatorics* **32** (2025).
- [11] Khanna, A., “Enumeration of Even Dimensional Partitions modulo 4”, *arXiv* (2025).
- [12] Khanna, A., “Counting Cores and Bar Cores: From Modular Forms to McKay numbers”, MS thesis (2022).
- [13] Macdonald, I., “On the Degrees of the Irreducible Representations of Symmetric Groups”, *Bulletin Of The London Mathematical Society* **3** (1971), pp. 189–192.

- [14] Macdonald, I., “On the Degrees of the Irreducible Representations of Finite Coxeter Groups”, *Journal Of The London Mathematical Society* **2.6** (1973), pp. 298–300.
- [15] Malle, G. & Späth, B., “Characters of odd degree”, *Annals Of Mathematics* **184** (2016), pp. 869–908.
- [16] McKay, J., “Irreducible representations of odd degree”, *Journal Of Algebra* **20** (1972), pp. 416–418.
- [17] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. (Published electronically at <https://oeis.org>, 2026)
- [18] Olsson, J., “Combinatorics and representations of finite groups”, *Vorlesungen aus dem Fachbereich Mathematik der Universität GH Essen* **20**, Universität Essen (1993).
- [19] Olsson, J., “McKay numbers and Heights of Characters”. *Mathematica Scandinavica*, **38** (1976), pp. 25–42.
- [20] Peluse, S. & Soundararajan, K., “Divisibility of character values of the symmetric group by prime powers”, *Algebra & Number Theory* **19** (2025), pp. 365–382.
- [21] Sagan, B., *The Symmetric Group (Graduate Texts In Mathematics)*, (2001).
- [22] Sloman, L. “After 20 Years, Math Couple Solves Major Group Theory Problem”, *Quanta Magazine*. (2025).

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