American Journal of Combinatorics Research Article
Volume 4 (2025), Pages 23–41

Automorphisms of the generalized cluster complex

Matthieu Josuat-Vergès

(Communicated by Bahattin Yildiz)

Abstract

We exhibit a dihedral symmetry in the generalized cluster complex defined by Fomin and Reading. Together with diagram symmetries, they generate the automorphism group of the complex. A consequence is a simple explicit formula for the order of this automorphism group.

1 Introduction

Fomin and Zelevinsky [8] defined the *cluster complex* associated to a finite-type cluster algebra: it is a simplicial complex having cluster variables as vertices and clusters as facets. Furthermore, the construction of Fomin and Zelevinsky involves a pair of maps τ_+ and τ_- that act as automorphisms of the cluster complex, generating a dihedral group of symmetry. In type A_{n-1}, the cluster complex can be described combinatorially as the complex of *dissections* of a regular $(n+2)$-gon (in particular, its facets are triangulations of this polygon), and the dihedral group of symmetries coincides with the apparent geometric symmetry of the $(n+2)$-gon. The automorphism group of the cluster complex (which is essentially the dihedral group generated by τ_+ and τ_-), and related automorphisms of finite-type cluster algebras, have been investigated in [1, 2, 5, 9].

The *generalized cluster complex* $\Gamma^{(m)}$ was introduced by Fomin and Reading [7], in the framework of finite-type Coxeter combinatorics. It depends on an integer parameter $m \geq 1$, in such a way that $m = 1$ (in crystallographic types) corresponds to the original definition of Fomin and Zelevinsky. There is no underlying cluster algebra, but there is a representation theoretic interpretation that lead to many developments such as higher cluster categories. We refer to [4, 6, 10, 11, 13]. In particular, Stump, Thomas, and Williams [10] give a thorough combinatorial treatment in [10]. In type A_{n-1}, the generalized cluster complex can be described via dissections of a $(mn+2)$-gon (and thus naturally embeds as a subcomplex of the cluster complex of type A_{mn-1}).

MSC2020: 05E18, 05E45; Keywords: Cluster complex, Finite Coxeter group, Automorphism group, Dihedral group, Symmetry
Received Aug 5, 2024; Revised Apr 3, 2025; Accepted Apr 4, 2025; Published Apr 7, 2025

The original definition of Fomin and Reading makes clear that the complex $\Gamma^{(m)}$ has a cyclic symmetry, via the action of an automorphism $\mathcal{R}$ (see Section 2). In the case $m = 1$, we have $\mathcal{R} = \tau_+\tau_-$. Here, we extend the dihedral symmetry by introducing involutive automorphisms $\mathcal{S}$ and $\mathcal{T}$ (see Section 5) that specialize in τ_+ and τ_- when $m = 1$ and satisfy $\mathcal{R}^m = \mathcal{S}\mathcal{T}$. This dihedral symmetry group is almost the full automorphism group of $\Gamma^{(m)}$, in the sense that we only need to add extra automorphisms coming from symmetry of the Coxeter graph. The precise statement is as follows.

Theorem 1.1. *Let W be a finite and irreducible Coxeter group, and let $\Gamma^{(m)}$ be the associated generalized cluster complex. Define two subgroups:*

- $\mathrm{Dih} \subset \mathrm{Aut}(\Gamma^{(m)})$, *called the* dihedral subgroup, *is generated by $\mathcal{R}$ and $\mathcal{S}$ (or $\mathcal{R}$ and $\mathcal{T}$).*
- $\mathrm{Diag} \subset \mathrm{Aut}(\Gamma^{(m)})$ *is the subgroup of* diagram automorphisms *(See Section 4).*

Then we have a semidirect product

$$\mathrm{Aut}(\Gamma^{(m)}) = \mathrm{Dih} \rtimes (\mathrm{Diag}\,/\langle \mathcal{C}\rangle),$$

where $\mathcal{C}$ is the canonical diagram automorphism (see Section 4.3), and

$$\left|\mathrm{Aut}(\Gamma^{(m)})\right| = (mh + 2)\omega \tag{1.1}$$

where h is the Coxeter number of W, and ω is the number of automorphisms of its Coxeter graph (i.e., $\omega = |\mathrm{Diag}|$).

This article is organized as follows:

- In Section 2, we review the definition and properties of $\Gamma^{(m)}$.
- Sections 3 and 4 contain elementary facts about the reducible case and diagram automorphisms, respectively. The more technical Section 5 introduces the involutive automorphisms $\mathcal{S}$ and $\mathcal{T}$.
- The proof of Theorem 1.1 is contained in Sections 6, 7, 8. By a global induction hypothesis, we will assume that the statements in these three sections hold in rank $< n$ where n is the rank of W.

There are a few final remarks in Section 9.

2 Review of the generalized cluster complex

Let Φ be a finite-type root system, associated to the finite Coxeter group W. Let $\Phi = \Phi_+ \uplus \Phi_-$ be a decomposition in positive and negative roots. Let $\Delta \subset \Phi_+$ be the set of simple roots, S the set of simple reflections, T the set of reflections. For each root $\alpha \in \Phi$, the associated reflection is denoted $t_\alpha \in T$. For $w \in W$, let $\mathrm{Supp}(w) \subset S$ denote the *support* of w, *i.e.*, the set of simple reflections that appear in reduced words for w. The required background on reflection groups is rather small, as our work is mostly based on the combinatorial properties of $\Gamma^{(m)}$ that we review here.

Definition 2.1. A *colored root* is a pair $(\alpha, i) \in \Phi \times [\![1, m]\!]$. It is denoted α^i and i is called its *color*. It is *almost-positive* if $\alpha \in \Phi_+$, or $-\alpha \in \Delta$ and $i = 1$. The set of almost-positive colored roots is denoted $\Phi^{(m)}_{\geq -1}$.

This set $\Phi^{(m)}_{\geq -1}$ is the vertex set of the generalized cluster complex $\Gamma^{(m)}$. This is a simplicial complex with many interesting geometric and enumerative features, see [6, 7, 10, 12]. Besides the original definition by Fomin and Reading [7], a review which is sufficient for our purpose has been given in [6, Section 6]. We give an outline and refer to *loc. cit.* for any further information.

Before going into the technical definitions, let us just mention the combinatorial construction of the generalized cluster complex. In type A_{n-1} (W is the symmetric group $\mathfrak{S}_n$), the Coxeter number is $h = n$. Consider a regular $(mn + 2)$-gon. Recall that a *dissection* of a polygon is a set of pairwise noncrossing diagonals (note that a side of the polygon is *not* a diagonal). Such a dissection divides the polygon into smaller polygons, for example a triangulation is a dissection where all inner polygons are triangles. The generalized cluster complex $\Gamma^{(m)}$ of type A_{n-1} can be identified to the complex where:

- vertices are diagonals that divide the $(mn + 2)$-gon into a $(kn + 2)$-gon and a $((m - k)n + 2)$-gon, with $1 \leq k \leq m - 1$,
- facets are $(m + 2)$-angulations, *i.e.*, dissections such that all inner polygon are $(k + 2)$-gon.

Moreover, the faces of the complex $\Gamma^{(m)}$ in type B_n correspond to centrally-symmetric faces of the complex of type A_{2n-1}. We refer to [7] for more details (for example, the correspondence between $\Phi^{(m)}_{\geq -1}$ and diagonals of the polygon). The point is that all the combinatorial constructions makes clear that the dihedral symmetries of the $(mn + 2)$-gon induce symmetries of the complex $\Gamma^{(m)}$.

2.1 The compatibility relation

Fomin and Reading originally defined $\Gamma^{(m)}$ as the flag complex associated with a (symmetric) binary relation on $\Phi^{(m)}_{\geq -1}$, called *compatibility* of almost-positive colored roots. This is relevant in the present context, since the automorphism group of a flag complex is clearly the automorphism group of its 1-skeleton.

There exists a black/white decomposition $\Delta = \Delta_\bullet \uplus \Delta_\circ$ where $\Delta_\bullet$, respectively $\Delta_\circ$, contains pairwise orthogonal roots. The *bipartite Coxeter element* c is defined by:

$$c_\bullet = \prod_{\alpha \in \Delta_\bullet} t_\alpha, \quad c_\circ = \prod_{\alpha \in \Delta_\circ} t_\alpha, \quad \text{and} \quad c = c_\bullet c_\circ.$$

It is also possible to consider an arbitrary *standard Coxeter element* (see [10]), but the complexes obtained this way are all isomorphic. So, we stick to the bipartite Coxeter element as in [7].

Definition 2.2 ([7]). The self-bijection $\mathcal{R} : \Phi^{(m)}_{\geq -1} \to \Phi^{(m)}_{\geq -1}$ is defined by

$$\mathcal{R}(\alpha^i) := \begin{cases} \alpha^{i+1} & \text{if } \alpha \in \Phi_+ \text{ and } i < m, \\ (-\alpha)^1 & \text{if } \alpha \in \Delta_\circ \text{ and } i = m, \text{ or } -\alpha \in \Delta_\bullet \text{ and } i = 1, \\ c(\alpha)^1 & \text{if } \alpha \in \Phi_+ \backslash \Delta_\circ \text{ and } i = m, \text{ or } -\alpha \in \Delta_\circ \text{ and } i = 1. \end{cases}$$

Let w_0 denote the longest element of W.

Remark 2.3. The longest element w_0 sends Δ to $-\Delta$, more precisely $\rho \mapsto -w_0(\rho)$ is an automorphism of the Coxeter diagram of W (see also Section 4.3 for the explicit description of this automorphism in the irreducible types).

Lemma 2.4 ([7]). *Let $X \subset \Phi^{(m)}_{\geq -1}$ be an $\mathcal{R}$-orbit. We have either:*

- *$\#X = \frac{mh+2}{2}$, and X contains exactly one element of the form $-\rho^1$ (with $\rho \in \Delta$).*
- *$\#X = mh + 2$, and X contains exactly two elements of the form $-\rho^1$ (with $\rho \in \Delta$). Moreover, these two elements have the form $-\rho^1$ and $w_0(\rho)^1$.*

The order of $\mathcal{R}$ is given by

$$\frac{|\mathcal{R}|}{|w_0|} = \frac{mh+2}{2}. \tag{2.1}$$

Definition 2.5 ([7]). The *compatibility relation* is a symmetric binary relation on $\Phi^{(m)}_{\geq -1}$, uniquely defined by the two conditions:

- $\parallel$ is preserved by $\mathcal{R}$ (*i.e.*, we have $\alpha \parallel \beta$ if and only if $\mathcal{R}(\alpha) \parallel \mathcal{R}(\beta)$),
- if $\alpha \in \Delta$, we have $-\alpha^1 \parallel \beta^j$ if and only if $t_\alpha \notin \mathrm{Supp}(t_\beta)$.

See [7] for details on the existence and unicity.

Definition 2.6 ([7]). The *generalized cluster complex* $\Gamma^{(m)}$ is defined as the flag complex associated with the binary relation $\parallel$. This means that a subset of $\Phi^{(m)}_{\geq -1}$ is a face of $\Gamma^{(m)}$ iff its elements are pairwise compatible.

We will use the following rules, which make the compatibility relation more explicit. The *absolute order* on W is denoted $\leq$. The elements of w below c in the absolute order are called *noncrossing partitions*, and are closely related to the cluster complex (see [3, 12]). Let $\alpha, \beta \in \Delta$, and $1 \leq i < j \leq m$. We have:

$$-\alpha^1 \parallel \beta^i \iff t_\alpha \notin \mathrm{Supp}(t_\beta), \tag{2.2}$$

$$\alpha^i \parallel \beta^i \iff \langle \alpha | \beta \rangle \geq 0, \text{ and } t_\alpha t_\beta \leq c \text{ or } t_\beta t_\alpha \leq c, \tag{2.3}$$

$$\alpha^i \parallel \beta^j \iff t_\alpha t_\beta \leq c. \tag{2.4}$$

One way to see this is to use the total order on $\Phi^{(m)}_{\geq -1}$ in the next section, see also [6, Section 6].

2.2 Reflection ordering

An alternative characterization of faces in $\Gamma^{(m)}$ has been given by Tzanaki [12]. It makes a connection with factorization of the Coxeter element. The idea is to use a total order on $\Phi^{(m)}_{\geq -1}$, akin to Steinberg's indexing of Φ used by Brady and Watt [3]. Recall that the Steinberg ordering of Φ is given by indexing roots $(\alpha_i)_{1\leq i\leq nh}$ with the conditions:

- $-\Delta_\circ = \{\alpha_i \;:\; 1\leq i\leq r\}$ (where $r = \#\Delta_\circ$)
- $\Delta_\bullet = \{\alpha_i \;:\; r+1\leq i\leq n\}$
- $c(\alpha_i) = \alpha_{i+n}$ (where indices are taken modulo nh).

Almost-positive roots are the roots $(\alpha_i)_{1\leq i\leq \frac{nh}{2}+n}$, and the indexing gives a total order on $\Phi_{\geq -1}$. Brady and Watt showed that facets of the cluster complex correspond to decreasing factorizations of the Coxeter element [3, Section 8]. Also, there is a decomposition:

$$\begin{aligned}\Phi_{\geq -1} = -\Delta_\circ \uplus \Delta_\bullet \uplus c(-\Delta_\circ) \uplus c(\Delta_\bullet) \uplus c^2(-\Delta_\circ) \uplus c^2(\Delta_\bullet) \uplus \cdots \\ \cdots \uplus c^{-2}(-\Delta_\bullet) \uplus c^{-1}(\Delta_\circ) \uplus c^{-1}(-\Delta_\bullet) \uplus \Delta_\circ \uplus -\Delta_\bullet,\end{aligned} \tag{2.5}$$

which coarsens the total order in the sense that each block contain consecutive elements and the blocks are written increasingly.

Tzanaki's generalization is a total order $\prec$ on $\Phi^{(m)}_{\geq -1}$ given as follows. Rather than a total order, it is helpful to think as $\Phi^{(m)}_{\geq -1}$ as a union of $mh+2$ blocks, generalizing (2.5). Denote X^i for $\{\alpha^i \;:\; \alpha\in X\} \subset \Phi^{(m)}_{\geq -1}$ if $X\subset\Phi$. We have:

$$\begin{aligned}\Phi^{(m)}_{\geq -1} = -\Delta^1_\circ \uplus \Delta^m_\bullet \uplus \cdots \uplus \Delta^m_\circ \uplus \Delta^{m-1}_\bullet \uplus \cdots \uplus \Delta^{m-1}_\circ \uplus \quad \cdots \\ \cdots \quad \uplus \Delta^2_\bullet \uplus \cdots \uplus \Delta^2_\circ \uplus \Delta^1_\bullet \uplus \cdots \uplus \Delta^1_\circ \uplus -\Delta^1_\bullet,\end{aligned} \tag{2.6}$$

where it is understood that $\Delta^i_\bullet \uplus \cdots \uplus \Delta^i_\circ$ contains h terms, as in (2.5) except the first and last.

The total order on $\Phi^{(m)}_{\geq -1}$ is such that the decomposition in (2.6) is a coarsening (as above), and each block is ordered via Steinberg's ordering.

Proposition 2.7 ([12])**.** Let f be a tuple of n elements of $\Phi^{(m)}_{\geq -1}$, indexed $f = (\gamma_i^{k_i})_{1\leq i\leq n}$ such that $\gamma_1^{k_1} \succ \cdots \succ \gamma_n^{k_n}$. Then f is a facet of $\Gamma^{(m)}$ iff $c = t_{\gamma_1}\cdots t_{\gamma_n}$.

Remark 2.8. We will see that the decomposition in (2.6) is well-behaved with respect to the automorphisms of $\Gamma^{(m)}$. For example, $\mathcal{R}$ sends a block of index i to the block of index $i-m$ (where indices are taken modulo $mh+2$). We will see that $\mathcal{S}$ fixes $-\Delta^1_\circ$ and reverses the order on the other blocks. Similarly, $\mathcal{T}$ fixes $-\Delta^1_\bullet$ and reverses the order on the other blocks. Note that there might exist nontrivial automorphisms that act trivially on the decomposition in (2.6) (these are the even diagram automorphisms, see Section 4).

A direct consequence of the previous proposition is the following:

Lemma 2.9. *If $\alpha^k \parallel \beta^\ell$, we have $t_\alpha \neq t_\beta$, and $t_\alpha t_\beta \leq c$ or $t_\beta t_\alpha \leq c$. (Again, $\leq$ is the absolute order on W.) More precisely, $\alpha^k \parallel \beta^\ell$ and $\alpha^k \prec \beta^\ell$ imply $t_\beta t_\alpha \leq c$.*

See [3] for more on the absolute order.

2.3 The other bipartite Coxeter element

Note that the definition of $\Gamma^{(m)}$ depends on the choice of $\Delta_\bullet$ and $\Delta_\circ$. The exchange of $\bullet$ and $\circ$ gives an isomorphic complex and we make this explicit here.

Denote by $\check{\Gamma}^{(m)}$ the complex defined similar to $\Gamma^{(m)}$ but exchanging the roles of $\Delta_\bullet$ and $\Delta_\circ$. Similarly, we denote $\check{\|}$ and $\check{\mathcal{R}}$ the analog of $\|$ and $\mathcal{R}$.

Proposition 2.10. The involutive self-map $\iota : \Phi^{(m)}_{\geq -1} \to \Phi^{(m)}_{\geq -1}$ defined by

$$\iota(\alpha^i) = \begin{cases} \alpha^i & \text{if } \alpha \in -\Delta, \\ \alpha^{m+1-i} & \text{if } \alpha \in \Phi_+ \end{cases}$$

induces an isomorphism $\iota : \Gamma^{(m)} \to \check{\Gamma}^{(m)}$. Moreover, we have: $\iota \mathcal{R} \iota = \check{\mathcal{R}}$.

The proof is straightforward and omitted.

2.4 Links

A useful consequence of the definition of the compatibility relation is the following. Let $\alpha \in \Delta$, and W_α the maximal standard parabolic subgroup of W obtained by removing α from the Coxeter graph of W. The *link* of a face $f \in \Gamma^{(m)}$ is by definition

$$\mathrm{Link}(f) = \Big\{ f' \ : \ f \cap f' = \varnothing \text{ and } f \cup f' \in \Gamma^{(m)} \Big\}.$$

Note that $\mathrm{Link}(f)$ is itself a simplicial complex.

Proposition 2.11. Let $\rho \in \Delta$. We have $\mathrm{Link}(\{-\rho^1\}) \simeq \Gamma^{(m)}(W_\rho)$.

It also follows that the link of any face in $f \in \Gamma^{(m)}$ is isomorphic to $\Gamma^{(m)}(P)$ where P is a standard parabolic subgroup of W. A precise way to give the parabolic subgroup P in terms of f is given in [6, Proposition 2.20] (this is not needed in the present work).

A concrete consequence of this property of links is the following:

Theorem 2.12. *Let W and W' be finite Coxeter groups. If $\Gamma^{(m)}(W)$ and $\Gamma^{(m')}(W')$ are isomorphic, then $m = m'$ and W and W' are Coxeter-isomorphic.*

Proof. We show that m and the Coxeter graph of W can be recovered from $\Gamma^{(m)}(W)$. The link of a 1-codimensional face in $\Gamma^{(m)}(W)$ is $\Gamma^{(m)}(A_1)$ (since A_1 is the unique Coxeter group of rank 1), which consists in $m+1$ isolated vertices. So m can be recovered from $\Gamma^{(m)}(W)$. Note also that the rank of W is $1 + \dim(\Gamma^{(m)})$.

We then proceed by induction on n, the rank of W. The case $n = 1$ is trivial, and the case $n = 2$ is settled by noting that the number of facets of $\Gamma^{(m)}(I_2(k))$ strictly increases with k (this number is known as the Fuß-Catalan number, see [7] for the exact formula).

We need the following: if $n \geq 3$, the Coxeter graph of W is uniquely characterized by the collection of Coxeter graphs of W_α ($\alpha \in \Delta$). It is straightforward to check this. As the links of vertices of $\Gamma^{(m)}$ provide this collection (by induction hypothesis), this completes the proof. □

3 The reducible case

We examine the situation where W can be decomposed as a product

$$W = W_1 \times \cdots \times W_r.$$

According to [7], we have

$$\Gamma^{(m)}(W) = \Gamma^{(m)}(W_1) \star \cdots \star \Gamma^{(m)}(W_r) \tag{3.1}$$

where $\star$ is the join operation on simplicial complexes. Concretely, this means that each face $f \in \Gamma^{(m)}(W)$ can be written

$$f = (f_1, \dots, f_r) \tag{3.2}$$

where f_i is a face of $\Gamma^{(m)}(W_i)$, moreover $\dim(f) = \sum_{i=1}^{r} \dim(f_i)$. In particular, at the level of vertex sets we have

$$\Phi^{(m)}_{\geq -1}(W) = \biguplus_{i=1}^{r} \Phi^{(m)}_{\geq -1}(W_i). \tag{3.3}$$

Definition 3.1. An automorphism $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$ is *monomial* if the partition of the vertex set in (3.3) is stabilized: for any i, there exists j such that we have $\mathcal{F}\big(\Phi^{(m)}_{\geq -1}(W_i)\big) = \Phi^{(m)}_{\geq -1}(W_j)$.

The terminology is an analogy with monomial matrices.

Proposition 3.2. Every element $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$ is monomial.

The idea is to give a combinatorial criterion to characterize when two vertices are in the same block of the partition.

Lemma 3.3. *Let $\alpha^k \in \Phi^{(m)}_{\geq -1}(W_i)$ and $\beta^\ell \in \Phi^{(m)}_{\geq -1}(W_j)$ with $1 \leq i, j \leq r$. The following conditions are equivalent:*

- $i \neq j$,
- $\alpha^k \parallel \beta^\ell$, *and for all* $\gamma^p \in \Phi^{(m)}_{\geq -1}$ *we have* $\alpha^k \parallel \gamma^p$ *or* $\beta^\ell \parallel \gamma^p$.

Proof. First, assume that $i \neq j$. We have $\alpha^k \parallel \beta^\ell$ by (3.1) and by definition of the join. Let $\gamma^p \in \Phi^{(m)}_{\geq -1}(W_\ell)$. We have $\alpha^k \parallel \gamma^p$ if $i \neq \ell$ and $\beta^\ell \parallel \gamma^p$ if $j \neq \ell$. Since $i \neq j$, at least one of $\alpha^k \parallel \gamma^p$ or $\beta^\ell \parallel \gamma^p$ thus holds. Therefore the second condition in the lemma holds.

Now, assume $i = j$. If $\alpha^k \nparallel \beta^\ell$, the second condition in the lemma does not hold. So, we assume $\alpha^k \parallel \beta^\ell$. It remains to prove the following: there exists γ^p such that $\alpha^k \nparallel \gamma^p$ and $\beta^\ell \nparallel \gamma^p$. By using the map $\mathcal{R}$ on the irreducible factor W_i and the invariance of compatibility, we can assume that $\alpha \in -\Delta$ (and $k = 1$). The construction of γ^p is as follows.

- If $\beta \in -\Delta$ (and $\ell = 1$), let $\gamma \in \Phi_+$ be such that $t_\alpha \in \mathrm{Supp}(t_\gamma)$ and $t_\beta \in \mathrm{Supp}(t_\gamma)$ (it exists because t_α and t_β are both in W_i which is irreducible, and a finite irreducible Coxeter group has reflections with full support). The color p can be anything.

- Otherwise, let $P \subset W_i$ be a standard parabolic subgroup such that P is irreducible, $t_\alpha \in P$, $P \cap \text{Supp}(t_\beta) = \varnothing$, and P contains a neighbor of $\text{Supp}(t_\beta)$ in the Coxeter graph. (It exists because t_α and t_β are both in W_i which is irreducible, and these properties easily translate into properties of the corresponding subset of S or Δ. We used the fact that $t_\alpha \notin \text{Supp}(t_\beta)$, which holds since $\alpha^k \parallel \beta^\ell$). Let $\gamma \in \Delta$ such that $\text{Supp}(t_\gamma) = P \cap S$, and p is arbitrary. Now, we have:
  - $\alpha^k \nparallel \gamma^\ell$ holds via (2.2), since $t_\alpha \in P \cap S = \text{Supp}(t_\gamma)$.
  - $\beta^\ell \nparallel \gamma^\ell$ holds via (2.4), since $\langle \beta | \gamma \rangle < 0$. This is obtained as follows. Write

$$\beta = \sum_{\rho \in \Delta,\, t_\rho \in \text{Supp}(t_\beta)} x_\rho \cdot \rho, \qquad \gamma = \sum_{\rho \in \Delta,\, t_\rho \in \text{Supp}(t_\gamma)} y_\rho \cdot \rho$$

    with $x_\rho > 0$ and $y_\rho > 0$. By expanding $\langle \beta | \gamma \rangle$, the unique nonzero term is $x_\sigma y_\tau \langle \sigma | \tau \rangle$ where σ and τ are neighbors in the Coxeter graph, so that $\langle \sigma | \tau \rangle < 0$.

In both cases, we have found γ^p with the desired properties. This shows that if $i = j$, the second condition in the lemma doesn't hold. □

Proof of Proposition 3.2. Let $\alpha \in \Phi^{(m)}_{\geq -1}(W_i)$ and $\beta \in \Phi^{(m)}_{\geq -1}(W_j)$ with $1 \leq i, j \leq k$, and $\mathcal{F} \in \text{Aut}(\Gamma^{(m)})$. Because the second condition in the previous lemma is invariant under $\mathcal{F}$, α and β are in the same block of the vertex set partition if and only if $\mathcal{F}(\alpha)$ and $\mathcal{F}(\beta)$ have the same property. So, $\mathcal{F}$ stabilizes the vertex set partition in (3.3), which means that it is a monomial automorphism. □

Let us describe one particular application of the previous proposition.

Remark 3.4. Assume now that W is irreducible, let $\alpha \in \Delta$, and let $W' = W_\alpha$ be the maximal standard parabolic subgroup of W obtained by removing α in the Coxeter graph of W. Let $\mathcal{F} \in \text{Aut}(\Gamma^{(m)}(W))$. If $\mathcal{F}(-\alpha^1) = -\alpha^1$, the restriction of $\mathcal{F}$ to the link of $-\alpha^1$ is also an automorphism that we denote $\mathcal{F}' \in \text{Aut}(\Gamma^{(m)}(W'))$. Because every automorphism of $\Gamma^{(m)}(W')$ is monomial, there is a type-preserving permutation of the irreducible factors of W' underlying $\mathcal{F}$. This will be helpful in understanding the stabilizer of $-\alpha^1$ in $\text{Aut}(\Gamma^{(m)})$.

4 Diagram automorphism

Let $\mathcal{D} : \Delta \to \Delta$ be an automorphism of the Coxeter graph of W. There is an induced automorphism $W \to W$, and an induced self-bijection $\Phi \to \Phi$. We keep the notation $\mathcal{D}$ for these induced maps.

4.1 Even diagram automorphisms

First, we assume that $\Delta_\bullet$ and $\Delta_\circ$ are preserved by $\mathcal{D}$. A non-trivial such automorphism exists in types A_n with n odd, E_6, D_n with $n \geq 4$ (and it is unique except for type D_4 where the group of even diagram automorphism is the symmetric group $\mathfrak{S}_3$).

We extend $\mathcal{D}$ to $\Phi^{(m)}_{\geq -1}$ by $\mathcal{D}(\alpha^i) = \mathcal{D}(\alpha)^i$. It is straighforward to check that the map $\mathcal{D}$ preserves the compatibility relation, so that $\mathcal{D} \in \mathrm{Aut}(\Gamma^{(m)})$. It is called an *even diagram automorphism*. Moreover, we have $\mathcal{DR} = \mathcal{RD}$.

4.2 Odd diagram automorphisms

Now, assume that $\Delta_\bullet$ and $\Delta_\circ$ are exchanged by $\mathcal{D}$. A non-trivial such automorphism exists in types A_n with n even, F_4 and $I_2(k)$ (and it is unique).

We extend $\mathcal{D}$ to $\Phi^{(m)}_{\geq -1}$ by

$$\mathcal{D}(\alpha^i) = \begin{cases} \mathcal{D}(\alpha)^i & \text{if } \alpha \in -\Delta, \\ \mathcal{D}(\alpha)^{m+1-i} & \text{if } \alpha \in \Phi_+. \end{cases}$$

It is straightforward to check that this gives an automorphism $\mathcal{D} \in \mathrm{Aut}(\Gamma^{(m)})$, which is called an *odd diagram automorphism*. Moreover, we have $\mathcal{DR} = \mathcal{R}^{-1}\mathcal{D}$.

4.3 The canonical diagram automorphism

Let us recall standard facts of Coxeter theory. Conjugation by the longest element w_0 acts on simple reflections as a symmetry of the Coxeter diagram (see also Remark 2.3). It is the identity when all exponents are even, and it is the unique nontrivial symmetry of order 2 in other cases (A_n, D_n with n odd, E_6, $I_2(2k+1)$). If h is even, we have $c^{h/2} = w_0$. More precisely, for $\alpha \in \Delta$ we have $-c^{h/2}(\alpha) \in \Delta$ and it is the image of α under the symmetry of the Coxeter diagram.

Definition 4.1. The *canonical diagram automorphism* $\mathcal{C} \in \mathrm{Diag}$ is the element of Diag associated to the map $\Delta \to \Delta$, $\alpha \mapsto -w_0(\alpha)$.

Lemma 4.2. *If h is even, we have $\mathcal{R}^{(mh+2)/2} = \mathcal{C}$.*

Proof. By the definition of $\mathcal{R}$, we get

$$\mathcal{R}^m(\alpha^i) = \begin{cases} c(\alpha)^i & \text{if } \alpha \in \Phi_+ \backslash \Delta_\bullet, \\ -c(\alpha)^{i-1} & \text{if } \alpha \in \Delta_\circ \text{ and } i > 1, \\ (-\alpha)^1 & \text{if } \alpha \in \Delta_\circ \text{ and } i = 1, \\ c(\alpha)^m & \text{if } \alpha \in -\Delta_\circ \text{ and } i = 1, \\ (-\alpha)^m & \text{if } \alpha \in -\Delta_\bullet \text{ and } i = 1. \end{cases} \tag{4.1}$$

From this, it is straightforward to compute $\mathcal{R}^{mh/2}$ in the different cases, and show that $\mathcal{R}^{mh/2+1}$ is the map $\alpha^i \mapsto -c^{h/2}(\alpha)^i$. □

Proposition 4.3. We have $\mathcal{C} \in \mathrm{Dih}$.

Proof. Via the previous lemma, it remains only to consider the case where h is odd, *i.e.*, W has type A_{2n}. Using the combinatorial model of polygon dissections, the result is clear. □

Recall from the introduction that $\mathrm{Diag} \subset \mathrm{Aut}(\Gamma^{(m)})$ is defined as the subgroup of diagram automorphisms. To complete this section, note that Lemma 6.2 below characterizes elements of Diag as the automorphisms of $\Gamma^{(m)}$ that stabilize $-\Delta^1$ (setwise).

5 Involutive automorphisms

We define the two involutive automorphisms $\mathcal{S}$ and $\mathcal{T}$, as outlined in the introduction.

Definition 5.1. The self-map $\mathcal{S}$ on $\Phi^{(m)}_{\geq -1}$ is defined by:

$$\mathcal{S}(\alpha^i) = \begin{cases} \alpha^i & \text{if } \alpha \in -\Delta_\circ \text{ and } i = 1 \text{ (i)}, \\ (-\alpha)^m & \text{if } \alpha \in -\Delta_\bullet \text{ and } i = 1 \text{ (ii)}, \\ (-\alpha)^1 & \text{if } \alpha \in \Delta_\bullet \text{ and } i = m \text{ (ii')}, \\ \alpha^{m-i} & \text{if } \alpha \in \Delta_\bullet \text{ and } 1 \leq i \leq m-1 \text{ (iii)}, \\ c_\bullet(\alpha)^{m+1-i} & \text{if} \alpha \in \Phi_+ \backslash \Delta_\bullet \text{ (iv)}. \end{cases} \tag{5.1}$$

To gain some insight about this definition, we let the reader check how $\mathcal{S}$ acts on the vertex partition in (2.6) as explained in Remark 2.8. Another helpful verification is the following: in the combinatorial model of polygon dissections (in type A_n), this map acts geometrically as a reflection of the polygon and its diagonals.

Lemma 5.2. *We have $\mathcal{S}^2 = \mathcal{I}$.*

Proof. This is straightforward from the different cases in the definition and from $c_\bullet$ being an involution on $\Phi_+ \backslash \Delta_\bullet$. □

Proposition 5.3. The map $\mathcal{S}$ induces an automorphism of $\Gamma^{(m)}$.

Proof. We show that the map $\mathcal{S}$ sends a facet of $\Gamma^{(m)}$ to another facet, using the description in terms of reduced factorization of the Coxeter element (Section 2.2).

Let $f = (\alpha_i^{k_i})_{1 \leq i \leq n}$ be a facet of $\Gamma^{(m)}$, indexed as in Proposition 2.7, so that $c = t_{\alpha_1} \cdots t_{\alpha_n}$. The *canonical factorization* of c associated with a facet of $\Gamma^{(m)}$ is:

$$c = w_\bullet w_1 \cdots w_m w_\circ, \tag{5.2}$$

defined as a coarsening of $t_{\alpha_1} \cdots t_{\alpha_n}$ where $w_\bullet$ (respectively, w_j, $w_\circ$) contains the factors t_{α_i} such that α_i is in $-\Delta_\bullet$ (respectively, Φ^j_+, $-\Delta_\circ$). We further refine this by writing

$$w_j = w_{j,2} w_{j,1}$$

where $w_{j,1}$ contains all factors t_{α_i} with $\alpha_i \in \Delta_\bullet$.

It is helpful to first assume that $f' := \mathcal{S}(f)$ is also a facet of $\Gamma^{(m)}$, and compute what should be the associated canonical factorization. Denote it:

$$c = w'_\bullet w'_1 \cdots w'_m w'_\circ,$$

and again we write

$$w'_i = w'_{i,2} w'_{i,1}$$

where $w'_{i,1}$ contains all factors t_α with $\alpha \in \Delta_\bullet$. Now, the definition of $\mathcal{S}$ gives necessary relations between all the factors we just defined:

$$\begin{aligned}
w'_\circ &= w_\circ \text{ (from (5.1), case (i))},\\
w'_{i,1} &= w_{m-i,1} \text{ where } 1 \le i \le m-1 \text{ (from (5.1), case (iii))},\\
w'_\bullet &= w_{m,1} \text{ (from (5.1), case (ii))},\\
w'_{m,1} &= w_\bullet \text{ (from (5.1), case (ii'))},\\
w'_{i,2} &= c_\bullet w^{-1}_{m+1-i,2} c_\bullet \text{ where } 1 \le i \le m \text{ (from (5.1), case (iv))}.
\end{aligned}$$

About the latter equality, let us explain why we need to take the inverse. Let $\alpha^i_1, \dots, \alpha^i_k \in \Phi^{(m)}_{\ge -1}$ be the vertices of f that contributes to the factors of $w_{i,2}$, ordered so that $w_{i,2} = t_{\alpha_1} \cdots t_{\alpha_k}$ (*i.e.*, decreasingly with respect to $\prec$). Let $\alpha'_i = c_\bullet(\alpha_i)$, so that $t_{\alpha'_i} = c_\bullet t_{\alpha_i} c_\bullet$. Note that $w_{i,2} \le c$ implies $c_\bullet w_{i,2} c_\bullet \le c_\bullet c c_\bullet = c^{-1}$, so that $c_\bullet w^{-1}_{i,2} c_\bullet \le c$ (it is well-known and easy to see from the definition that the absolute order is invariant under conjugation). The way to order the elements $t_{\alpha'_i}$ and get an element below c in the absolute order is thus $c_\bullet w^{-1}_{i,2} c_\bullet = t_{\alpha'_k} \cdots t_{\alpha'_1}$.

Now, let us check that these are indeed the factors of a factorization of c:

$$\begin{aligned}
w'_\bullet w'_1 \cdots w'_m w'_\circ &= w'_\bullet w'_{1,2} w'_{1,1} \cdots w'_{m,2} w'_{m,1} w'_\circ\\
&= w_{m,1} \cdot c_\bullet w^{-1}_{m,2} c_\bullet \cdot w_{m-1,1} \cdot c_\bullet w^{-1}_{m-1,2} c_\bullet \cdot w_{m-2,1} \cdots c_\bullet w^{-1}_{1,2} c_\bullet \cdot w_\bullet w_\circ.
\end{aligned}$$

Because $c_\bullet$ commutes with $w_{i,1}$, this gives:

$$\begin{aligned}
w'_\bullet w'_1 \cdots w'_m w'_\circ &= w_{m,1} \cdot c_\bullet \cdot w^{-1}_{m,2} w_{m-1,1} w^{-1}_{m-1,2} w_{m-2,1} \cdots w^{-1}_{1,2} c_\bullet \cdot w_\bullet w_\circ\\
&= c_\bullet (w_1 \cdots w_m)^{-1} \cdot c_\bullet w_\bullet w_\circ\\
&= c_\bullet (w_\circ c^{-1} w_\bullet) \cdot c_\bullet w_\bullet w_\circ = c_\bullet w_\circ c_\circ c_\bullet w_\bullet c_\bullet w_\bullet w_\circ = c_\bullet w_\circ c_\circ w_\circ = c.
\end{aligned}$$

The previous computations prove the proposition. Indeed, if $f = \{\alpha^{i_1}_1, \dots, \alpha^{i_n}_n\}$ and $f' = \{\mathcal{S}(\alpha^{i_1}_1), \dots, \mathcal{S}(\alpha^{i_n}_n)\}$, the factorization $c = w'_\bullet w'_1 \cdots w'_m w'_\circ$ can be refined as a reflection factorization that proves $f' \in \Gamma^{(m)}$ via Proposition 2.7.

An alternative proof would be check that $\mathcal{S}$ preserves the compatibility relation $\|$. This is straightforward, although a bit long because of the various cases to consider. □

Lemma 5.4. *We have $\mathcal{SRS} = \mathcal{R}^{-1}$.*

Proof. We make the composition $\mathcal{SR}$ explicit. First,

$$\mathcal{SR}(\alpha^i) = \begin{cases} \mathcal{S}(\alpha^{i+1}) = \alpha^{m-i-1} & \text{if } \alpha \in \Delta_\bullet \text{ and } 1 \le i \le m-2,\\ \mathcal{S}(\alpha^{i+1}) = c_\bullet(\alpha)^{m-i} & \text{if } \alpha \in \Phi_+ \backslash \Delta_\bullet \text{ and } 1 \le i \le m-1, \end{cases}$$

so $\mathcal{SR}$ is an involution on the elements considered in each case. Second,

$$\mathcal{SR}(\alpha^i) = \begin{cases} \mathcal{S}((\alpha)^1) = (-\alpha)^1 & \text{if } \alpha \in \Delta_\circ \text{ and } i = m,\\ \mathcal{S}(c(\alpha)^1) = \mathcal{S}(c_\bullet(-\alpha)^1) = (-\alpha)^m & \text{if } \alpha \in -\Delta_\circ \text{ and } i = 1, \end{cases}$$

and

$$\mathcal{SR}(\alpha^i) = \begin{cases} \mathcal{S}((-\alpha)^1) = (-\alpha)^{m-1} & \text{if } \alpha \in -\Delta_\bullet \text{ and } i = 1, \\ \mathcal{S}(\alpha^m) = (-\alpha)^1 & \text{if } \alpha \in \Delta_\bullet \text{ and } i = m-1, \end{cases}$$

so $\mathcal{SR}$ is also an involution on these elements. Eventually, we have:

$$\mathcal{SR}(\alpha^i) = \mathcal{S}(c(\alpha)^1) = c_\circ(\alpha)^m \quad \text{if } \alpha \in \Phi_+ \backslash \Delta_\circ \text{ and } i = m.$$

(To check this, distinguish the cases $\alpha \in \Delta_\bullet$, $\alpha \in c_\circ(\Delta_\bullet)$, and $\alpha \in \Phi_+ \backslash (\Delta_\circ \cup \Delta_\bullet \cup c_\circ(\Delta_\bullet))$). It is now clear that $\mathcal{SR}$ is an involution. □

The previous lemma means that $\langle \mathcal{R}, \mathcal{S} \rangle \subset \mathrm{Aut}(\Gamma^{(m)})$ is a dihedral subgroup. Recall from the introduction that it is denoted Dih.

Definition 5.5. The self-map $\mathcal{T}$ on $\Phi^{(m)}_{\geq -1}$ is defined by:

$$\mathcal{T}(\alpha^i) = \begin{cases} \alpha^i & \text{if } \alpha \in -\Delta_\bullet \text{ and } i = 1 \text{ (i)}, \\ -\alpha^1 & \text{if } \alpha \in \pm\Delta_\circ \text{ and } i = 1 \text{ (ii)}, \\ \alpha^{m+2-i} & \text{if } \alpha \in \Delta_\circ \text{ and } 2 \leq i \leq m \text{ (iii)}, \\ c_\circ(\alpha)^{m+1-i} & \text{if } \alpha \in \Phi_+ \backslash \Delta_\circ \text{ (iv)}. \end{cases} \tag{5.3}$$

Proposition 5.6. We have $\mathcal{T} \in \mathrm{Aut}(\Gamma^{(m)})$, moreover $\mathcal{T} = \iota \mathcal{S} \iota$ (*i.e.*, $\check{\mathcal{S}} = \mathcal{T}$ and $\check{\mathcal{T}} = \mathcal{S}$).

The proof is straightforward, by black/white symmetry.

Remark 5.7. When $m = 1$, these maps $\mathcal{S}$ and $\mathcal{T}$ are those considered by Fomin and Zelevinsky in [8]. One can check that $\mathcal{ST} = \mathcal{R}^m$ (the map $\mathcal{R}^m$ is explicit in (4.1)). The two maps $\mathcal{S}$ and $\mathcal{T}$ are thus generators of Dih if m is relatively prime with the order of $\mathcal{R}$. This happens in the following two situations:

- the order of $\mathcal{R}$ is $mh + 2$ and m is relatively prime to $mh + 2$ (i.e., m is odd).
- the order of $\mathcal{R}$ is $\frac{mh+2}{2}$ (h is even in this case, so that $m\frac{h}{2} + 1$ is relatively prime to m).

Because this is not exhaustive, it is not possible to build the dihedral group of symmetries with $\mathcal{S}$ and $\mathcal{T}$ as generators, although they look like a natural set of generators.

6 Stabilizer of a pair of vertices

Here and in the following sections, we take $\alpha, \beta \in \Delta$ such that β is the unique neighbor of some α in the Coxeter graph of W (α is a *leaf*). By symmetry (more precisely, using the isomorphism between $\Gamma^{(m)}$ and $\check{\Gamma}^{(m)}$ from Section 2.3), we can assume $\alpha \in \Delta_\bullet$ and $\beta \in \Delta_\circ$ in Proposition 6.1 and the other case follows. Let W_β be the maximal standard parabolic subgroup obtained by removing β in the Coxeter graph of W, and similarly for α.

Proposition 6.1. The (pointwise) stabilizer of $\{-\alpha^1, -\beta^1\}$ in $\mathrm{Aut}(\Gamma^{(m)})$ is generated by diagram automorphisms.

Lemma 6.2. *Assume that W is irreducible, and its rank is at least 2. If $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$ is such that $\mathcal{F}(-\rho^1) = -\rho^1$ for all $\rho \in \Delta$, then $\mathcal{F} = \mathcal{I}$.*

Proof. Consider the restriction of $\mathcal{F}$ to the link of a vertex $-\rho^1$, for $\rho \in \Delta$. Using an induction hypothesis, we can apply Proposition 7.1 to the maximal standard parabolic subgroup W_ρ, and find that the restriction of $\mathcal{F}$ on $\Gamma^{(m)}(W_\rho)$ is the identity. In particular, $\mathcal{F}(\rho^i)$ for $\rho \in \Delta$ and $1 \leq i \leq m$ (since ρ^i is in the link of $-\tau^1$ if $\tau \in \Delta$ and $\tau \neq \rho$).

To establish an induction showing that $\mathcal{F} = \mathcal{I}$, we consider the block decomposition in (2.6).

- We have shown that if the two neighbor blocks $-\Delta^1_\circ$ and $-\Delta^1_\bullet$ contain only fixed points, this is also the case (in particular) for the next blocks $\Delta^1_\circ$ and $\Delta^m_\bullet$.
- The automorphisms $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{T}$ can be used to send any pair of neighbor blocks in (2.6) to another one. Therefore, the previous property holds for any pair of consecutive blocks, not just $-\Delta^1_\circ$ and $-\Delta^1_\bullet$.

We conclude that every element of $\Phi^{(m)}_{\geq -1}$ is fixed by $\mathcal{F}$. □

In particular, the previous lemma settles the case of rank 2 in Proposition 6.1. We thus assume that the rank of W is at least 3 in the rest of this section.

Lemma 6.3. *Suppose that the rank of W is 3. Write $\Delta_\bullet = \{\alpha, \gamma\}$ and $\Delta_\circ = \{\beta\}$. There exists no automorphism $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$ such that $\mathcal{F}(-\alpha^1) = -\alpha^1$ and $\mathcal{F}(-\gamma^1) = \gamma^m$.*

Proof. By the finite-type classification, we have $t_\alpha t_\beta t_\alpha = t_\beta t_\alpha t_\beta$ or $t_\gamma t_\beta t_\gamma = t_\beta t_\gamma t_\beta$. The statement is symmetric in α and γ, because $\mathcal{F}(-\alpha^1) = -\alpha^1$ and $\mathcal{F}(-\gamma^1) = \gamma^m$ is equivalent to $\mathcal{S}\mathcal{F}(-\alpha^1) = \alpha^m$ and $\mathcal{S}\mathcal{F}(-\gamma^1) = -\gamma^1$. So, we assume $t_\gamma t_\beta t_\gamma = t_\beta t_\gamma t_\beta$.

Consider the 1-dimensional face $f = \{-\alpha^1, -\gamma^1\}$. We have:

$$\forall \rho^\ell \text{ vertex of } \mathrm{Link}(f),\ \mathrm{Link}(\{\rho^\ell\}) \simeq \Gamma^{(m)}(A_1 \times A_1). \tag{6.1}$$

Indeed, via (2.2), $\mathrm{Link}(f)$ is the 0-dimensional complex with vertices $\{-\beta^1, \beta^1, \dots, \beta^m\}$. These vertices are in the $\mathcal{R}$-orbit of $-\beta^1$. So their links are all isomorphic to the link of $-\beta^1$, which is $\Gamma^{(m)}(W_\beta) \simeq \Gamma^{(m)}(A_1 \times A_1)$.

Now, let $f' = \{-\alpha^1, \gamma^m\}$. Since the property in (6.1) is invariant under automorphisms, it remains only to show that this property doesn't hold with f' in place of f to conclude that no automorphism sends f to f'. To do this, let $\rho = t_\gamma(\beta)$. We check:

- $f' \cup \{\rho^m\} \in \Gamma^{(m)}$. The relation $-\alpha^1 \parallel \rho^m$ is clear via (2.2). The relation $\gamma^m \parallel \rho^m$ holds via (2.4) because: i) $t_\rho = t_\gamma t_\beta t_\gamma$ so that $t_\rho t_\gamma = t_\gamma t_\beta \leq c$, and ii) $\langle \rho | \gamma \rangle = \langle t_\gamma(\beta) | \gamma \rangle = -\langle \beta | \gamma \rangle > 0$.
- $\mathrm{Link}(\{\rho^m\}) \not\simeq \Gamma^{(m)}(A_1 \times A_1)$. We have $t_\rho = t_\gamma t_\beta t_\gamma = t_\beta t_\gamma t_\beta$ (by the assumption at the beginning of this proof). Since $c_\circ = t_\beta$, it follows that $c_\circ(\rho) = \gamma$, and $\mathcal{T}(\rho^m) = \gamma^1 = \mathcal{R}(-\gamma^1)$. So, $\{\rho^m\}$ and $\{-\gamma^1\}$ have isomorphic links. Since W is irreducible, $W_\gamma \not\simeq A_1 \times A_1$, and $\mathrm{Link}(\{\rho^m\}) \simeq \Gamma^{(m)}(W_\gamma) \not\simeq \Gamma^{(m)}(A_1 \times A_1)$ by Theorem 2.12.

So, (6.1) doesn't hold with f' in place of f. □

Proof of Proposition 6.1. Let $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$ with $\mathcal{F}(-\alpha^1) = -\alpha^1$ and $\mathcal{F}(-\beta^1) = -\beta^1$. Since $\mathcal{F}(-\alpha^1) = -\alpha^1$, the restriction of $\mathcal{F}$ to the link of $-\alpha^1$ gives an element $\mathcal{F}' \in \mathrm{Aut}(\Gamma^{(m)}(W_\alpha))$. Since $\{-\alpha^1, -\beta^1\} \in \Gamma^{(m)}$, $-\beta^1$ is in the link of $-\alpha^1$ and naturally identifies to a vertex $-\beta^1 \in \Gamma^{(m)}(W_\alpha)$ which is fixed by $\mathcal{F}'$. Using an induction hypothesis, we can apply Proposition 8.6.

Proposition 8.6 gives $\mathcal{F}' = \mathcal{S}'\mathcal{D}'$ or $\mathcal{F}' = \mathcal{D}'$, where $\mathcal{D}'$ is a diagram automorphism of $\Gamma^{(m)}(W_\alpha)$ fixing $-\beta^1$ (and $\mathcal{S}'$ is the restriction of $\mathcal{S}$ in $\mathrm{Aut}(\Gamma^{(m)}(W_\alpha))$). Note that $\mathcal{D}'$ is the restriction of a diagram automorphism $\mathcal{D}$ of $\Gamma^{(m)}(W)$ fixing $-\alpha^1$ and $-\beta^1$ (this is easily checked at the level of Coxeter graphs). Our goal is to show that $\mathcal{F} = \mathcal{D}$. Let $\mathcal{G} := \mathcal{F}\mathcal{D}^{-1}$.

By construction $\mathcal{G}$ fixes $-\alpha^1$ and $-\beta^1$, moreover its restriction on $\Gamma^{(m)}(W_\alpha)$, denoted $\mathcal{G}'$, is $\mathcal{S}'$ or the identity. By way of contradiction, assume that $\mathcal{G}' = \mathcal{S}'$. Let W_I be the standard parabolic subgroup of W with simple roots β and its neighbors in the Coxeter graph. If $\gamma \in \Delta$ is at distance 2 from β, the vertex $-\gamma^1$ is fixed by $\mathcal{S}'$ (because $\gamma \in \Delta_\circ$, just like β), and consequently it is also fixed by $\mathcal{G}$. It follows that the restriction of $\mathcal{G}$ gives an automorphism $\mathcal{H} \in \mathrm{Aut}(\Gamma^{(m)}(W_I))$. The situation can be summarized as follows.

- By definition of W_I, its Coxeter graph is a star centered at β. By the finite-type classification, either it has rank 3, or it is of type D_4.
- The vertices $-\alpha^1$ and $-\beta^1$ are fixed by $\mathcal{H} \in \mathrm{Aut}(\Gamma^{(m)}(W_I))$. The other vertices γ and possibly δ (if W_I has type D_4) are such that $\mathcal{H}(-\gamma^1) = \gamma^m$ and $\mathcal{H}(-\delta^1) = \delta^m$.

If the rank of W_I is 3, Lemma 6.3 gives a contradiction, so that such $\mathcal{H}$ doesn't exist. In the case where W_I has type D_4, consider the composition $\mathcal{S}\mathcal{H}$: $-\beta^1$, $-\gamma^1$, $-\delta^1$ are fixed, and the image of $-\alpha^1$ is α^m. By considering the link of $-\delta^1$, we see that the nonexistence of such $\mathcal{H}$ in type D_4 follows from the nonexistence in type A_3.

At this point, we have proved that $\mathcal{G}'$ is the identity. This proves that $\mathcal{G}$ fixes all the vertices $-\alpha^1$ for $\alpha \in \Delta$. By Lemma 6.2, it follows that $\mathcal{G} = \mathcal{I}$, so $\mathcal{F} = \mathcal{D}$. (Note that the assumption on the rank in Lemma 6.2 holds, because we assumed that the rank of W is at least 3.) □

7 Stabilizer of vertices

Recall that $\alpha \in \Delta_\bullet$ and $\beta \in \Delta_\circ$ are such that β is the unique neighbor of α in the Coxeter graph of W. The goal of this section is to describe the stabilizer of the vertex $-\beta^1$ in $\mathrm{Aut}(\Gamma^{(m)})$.

Proposition 7.1. Let $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$ be such that $\mathcal{F}(-\beta^1) = -\beta^1$. Then we have either $\mathcal{F} = \mathcal{D}$ or $\mathcal{F} = \mathcal{S}\mathcal{D}$, where $\mathcal{D}$ is an even diagram automorphism such that $\mathcal{D}(-\beta^1) = -\beta^1$.

Lemma 7.2. *Let $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$ be such that $\mathcal{F}(-\beta^1) = -\beta^1$. If $\mathcal{F}$ stabilizes the vertex set $\{-\alpha^1, \alpha^1, \dots, \alpha^m\}$, then it also stabilizes the vertex set $\{-\alpha^1, \alpha^m\}$.*

Proof. In this proof, a tuple of 4 elements of $\Phi^{(m)}_{\geq -1}$ is called a *square* if their induced subgraph in the compatibility graph (the 1-skeleton of $\Gamma^{(m)}$) is a cycle of length 4. The idea is to characterize the pair $\{-\alpha^1, \alpha^m\}$ among all pairs of elements in $\{-\alpha^1, \alpha^1, \dots, \alpha^m\}$ via certain squares containing $-\beta^1$. Note that $-\beta^1$ is compatible with elements in $\{-\alpha^1, \alpha^1, \dots, \alpha^m\}$ (via (2.2)), and $\{-\alpha^1, \alpha^1, \dots, \alpha^m\}$ doesn't contain a compatible pair (via Lemma 2.9). We check the following properties:

- $\{-\beta^1, -\alpha^1, \alpha^i\}$ can be completed to form a square if $i < m$. We take β^m as the fourth vertex. We have $\beta^m \parallel \alpha^i$ via (2.4) ($t_\alpha t_\beta \leq c$ can be seen by taking a subword of $c = c_\bullet c_\circ$).

- $\{-\beta^1, \alpha^i, \alpha^j\}$ can be completed to form a square if $1 \leq i < j \leq m$. Define $\gamma = t_\alpha(\beta)$, so that $t_\gamma t_\alpha = t_\alpha t_\beta \leq c$. If $i > 1$, this shows that we can take γ^1 as the fourth vertex of the square (via (2.4)). If $i = 1$, note that we also have $\langle \gamma | \alpha \rangle = -\langle \beta | \alpha \rangle > 0$ (we have $\langle \beta | \alpha \rangle < 0$ since α and β are neighbors in the Coxeter graph). Again, it follows that γ^1 can be chosen as the fourth vertex of a square (via (2.3)).

- $\{-\beta^1, -\alpha^1, \alpha^m\}$ cannot be completed to form a square. Assume otherwise, and let γ^i be the fourth vertex.
  - From $-\alpha^1 \parallel \gamma^i$, we get $t_\alpha t_\gamma \leq c$. From $\alpha^m \parallel \gamma^i$, we get $t_\gamma t_\alpha \leq c$. So $t_\alpha t_\gamma = t_\gamma t_\alpha$ (this follows from the fact that elements below c in the absolute order is a lattice, because $t_\alpha t_\gamma$ and $t_\gamma t_\alpha$ are rank 2 elements that covers the rank 1 elements t_α and t_γ). Thus, we get $\langle \alpha | \gamma \rangle = 0$.
  - Write $\gamma = \sum_{\rho \in \Delta} x_\rho \cdot \rho$ with real coefficients $x_\rho \geq 0$. Since γ^i is the fourth side of the square, we have $-\beta^1 \not\parallel \gamma^i$, so $x_\beta > 0$. Moreover $-\alpha^1 \parallel \gamma^i$ by definition of the square, so $c_\alpha = 0$. Now, $\langle \alpha, \gamma \rangle = \sum_{\rho \in \Delta} x_\rho \langle \alpha | \rho \rangle = x_\beta \langle \alpha | \beta \rangle < 0$ (since α and β are neighbors in the Coxeter graph), so that $\langle \alpha | \gamma \rangle < 0$.

  This gives a contradiction and completes this point.

Since squares are preserved by the automorphism $\mathcal{F}$, the triple $\{-\beta^1, -\alpha^1, \alpha^m\}$ is preserved as the unique one that cannot be completed to form a square among the triples considered above. The result follows. □

Proof of Proposition 7.1. The decomposition of W_β in irreducible components is written

$$W_\beta \simeq W_1 \times \cdots \times W_k.$$

Up to reindexing, assume that W_1 is the factor that contains the reflection t_α. By Proposition 3.2, the restriction of $\mathcal{F}$ on $\Gamma^{(m)}(W_\beta)$ permutes the irreducible factor of type A_1 in W_β. These irreducible factors of type A_1 correspond to leaves of the Coxeter graph that are neighbors of β. Clearly, any permutation of these leaves can be realized by an automorphism of the Coxeter graph that stabilizes β. So, there is a diagram automorphism $\mathcal{D}$ such that $\mathcal{FD}$ stabilizes $\Gamma^{(m)}(W_1) = \{-\alpha^1 \alpha^1, \dots, \alpha^m\}$ (setwise) and $-\beta^1$.

By Lemma 7.2, $\mathcal{FD}$ stabilizes $\{-\alpha^1, \alpha^m\}$. If $\mathcal{FD}$ stabilizes $-\alpha^1$ and $-\beta^1$, it follows from Lemma 6.1 that it is a diagram automorphism. Otherwise, $\mathcal{SFD}$ stabilizes $-\alpha^1$ and $-\beta^1$, and it follows from Lemma 6.1 that $\mathcal{SFD}$ is a diagram automorphism. This completes the proof. □

8 Automorphism groups

Lemma 8.1. *We have* $\mathrm{Aut}(\Gamma^{(m)}) = \mathrm{Dih} \cdot \mathrm{Diag}$.

Proof. Let $\mathcal{F} \in \mathrm{Aut}(\Gamma^{(m)})$, and $\alpha, \beta \in \Delta$ as in the previous sections. There is an integer i such that $\mathcal{R}^i\mathcal{F}(-\beta^1) \in -\Delta^1$.

If $\mathcal{R}^i\mathcal{F}(-\beta^1) = -\beta^1$, by Proposition 7.1 we obtain $\mathcal{R}^i\mathcal{F} = \mathcal{SD}$ or $\mathcal{R}^i\mathcal{F} = \mathcal{D}$, where $\mathcal{D} \in \mathrm{Diag}$. It follows $\mathcal{F} \in \mathrm{Dih} \cdot \mathrm{Diag}$.

In the general case, let $\gamma \in \Delta$ be such that $\beta \neq \gamma$ and $\mathcal{R}^i\mathcal{F}(-\beta^1) = -\gamma^1$. The links at $-\beta^1$ and $-\gamma^1$ are isomorphic since $\mathcal{R}^i\mathcal{F}$ is an automorphism of $\Gamma^{(m)}$. By Theorem 2.12, it follows that the Coxeter graphs of W_β and W_γ are isomorphic. So, there is a diagram automorphism $\mathcal{D}$ such that $\mathcal{D}(-\gamma^1) = -\beta^1$ (this is easily checked at the level of Coxeter graphs). So $\mathcal{G} = \mathcal{R}^i\mathcal{FD}$ is such that $\mathcal{G}(-\gamma^1) = -\gamma^1$. If $\gamma \in \Delta_\circ$, we obtain $\mathcal{G} \in \mathrm{Dih} \cdot \mathrm{Diag}$ by Proposition 7.1, and $\mathcal{F} \in \mathrm{Dih} \cdot \mathrm{Diag}$ follows. Otherwise, by black/white symmetry there is an analog statement (with $\mathcal{T}$ in place of $\mathcal{S}$), and we get $\mathcal{G} \in \mathrm{Dih} \cdot \mathrm{Diag}$ again. □

Lemma 8.2. *We have* $\mathrm{Dih} \cap \mathrm{Diag} = \langle \mathcal{C} \rangle$.

Proof. Note that $\mathcal{C} \in \mathrm{Dih} \cap \mathrm{Diag}$, by Proposition 4.3. It remains to check that there is no other diagram automorphism in Dih.

Suppose that $\mathcal{R}^i \in \mathrm{Diag}$ for some integer i. By Proposition 2.4, for any $\rho \in \Delta$ we have $\mathcal{R}^i(-\rho^1) = -\rho^1$ or $\mathcal{R}^i(-\rho^1) = w_0(\rho)^1$. It follows that $\mathcal{R}^i$ is either $\mathcal{I}$ or $\mathcal{C}$. It remains only to show that $\mathcal{R}^i\mathcal{S}$ cannot be a diagram automorphism other than $\mathcal{I}$ or $\mathcal{C}$.

Let us first consider the case where h is odd, *i.e.*, W has type A_{2n}. (It is treated separately because this is the only case where $\mathcal{C} = \mathcal{R}^i\mathcal{S}$ for some integer i.) Since $\mathrm{Diag} = \{\mathcal{I}, \mathcal{C}\}$, there is nothing to prove. We assume that h is even in the rest of this proof.

It remains to show $\mathcal{R}^i\mathcal{S} \notin \mathrm{Diag}$. Assume otherwise and let $\rho \in \Delta_\circ$. We get $\mathcal{R}^i(-\rho^1) = \mathcal{R}^i\mathcal{S}(-\rho^1) \in -\Delta^1$. Since h is even, by Lemma 4.2 it follows that $i \equiv 0 \mod \frac{mh+2}{2}$. But we then have $\mathcal{R}^i = \mathcal{I}$ or $\mathcal{R}^i = \mathcal{C}$, *i.e.*, $\mathcal{R}^i \in \mathrm{Diag}$. This is not possible since $\mathcal{S} \notin \mathrm{Diag}$. □

Lemma 8.3. *We have* $\mathrm{Dih} \triangleleft \mathrm{Aut}(\Gamma^{(m)})$.

Proof. If $|\mathrm{Diag}| = 2$, we have $[\,\mathrm{Aut}(\Gamma^{(m)}) : \mathrm{Dih}\,] \leq 2$ by the previous proposition and the result follows. This covers all cases except D_4.

If Diag contains only even diagram automorphisms, it commutes with Dih and the result follows from the previous lemma. This cover all cases except F_4, A_n (n even), $I_2(k)$. □

Theorem 8.4. *We have:*

$$\mathrm{Aut}(\Gamma^{(m)}) = \mathrm{Dih} \rtimes (\mathrm{Diag} / \langle \mathcal{C} \rangle).$$

Proof. First note that $\langle \mathcal{C} \rangle \triangleleft \mathrm{Diag}$, because its index is at most 2 (in all types except D_4) or because it is the trivial subgroup (to include D_4).

If $\langle \mathcal{C} \rangle$ is the trivial subgroup, the result follows from the previous lemmas. If $\mathrm{Diag} = \langle \mathcal{C} \rangle$, we get $\mathrm{Aut}(\Gamma^{(m)}) = \mathrm{Dih}$ and the result is clear. All cases are covered. □

Corollary 8.5. *Let* $\omega = |\mathrm{Diag}|$. *We have:*

$$\left|\operatorname{Aut}(\Gamma^{(m)})\right| = (mh+2)\omega.$$

Proof. By the previous theorem, we have

$$\left|\operatorname{Aut}(\Gamma^{(m)})\right| = \frac{|\mathrm{Dih}|}{|\mathcal{C}|}\omega.$$

Moreover, $|\mathrm{Dih}| = 2|\mathcal{R}|$. We have $|\mathcal{R}| = \frac{mh+2}{2}|w_0|$ by (2.1), and clearly $|w_0| = |\mathcal{C}|$. Putting this together completes the proof. □

The final step is the following generalization of Proposition 7.1, without restriction on the chosen vertex of the Coxeter diagram. Recall that it was used in the proof of Proposition 6.1 (where it was assumed to hold via an induction hypothesis).

Proposition 8.6. Let $\rho \in \Delta_\circ$ and $\mathcal{F} \in \operatorname{Aut}(\Gamma^{(m)})$ be such that $\mathcal{F}(-\rho^1) = -\rho^1$. Then we have either $\mathcal{F} = \mathcal{D}$ or $\mathcal{F} = \mathcal{SD}$, where $\mathcal{D}$ is an even diagram automorphism such that $\mathcal{D}(-\rho^1) = -\rho^1$. (And by black/white symmetry, there is a similar statement with $\Delta_\bullet$ and $\mathcal{T}$ in place of $\Delta_\circ$ and $\mathcal{S}$.)

Proof. We use the previous theorem and the orbit-stabilizer theorem. This suffices to show that the stabilizer contains only the listed elements. In most cases, the stabilizer has order 2 and it follows that it is $\{\mathcal{I}, \mathcal{S}\}$. We only give details about the other cases:

- If $\omega = 2$ and the orbit of $-\rho^1$ has cardinality $\frac{mh+2}{2}$: it means that the stabilizer has order 4, moreover $-\rho^1$ is fixed by the nontrivial element $\mathcal{D} \in \mathrm{Diag}$. It follows that the stabilizer is $\{\mathcal{I}, \mathcal{S}, \mathcal{D}, \mathcal{SD}\}$ (we have $\mathcal{SD} = \mathcal{DS}$ since $\mathcal{D}$ is even).

- In the the case of D_4, $\omega = 6$ and the orbit of $-\rho^1$ has cardinality $\frac{mh+2}{2}$. The stabilizer has cardinality 4 if ρ is a leaf of the Coxeter diagram and 12 if it is the central vertex. There are 2 elements of Diag stabilizing $-\rho^1$ in the first case, and 6 in the second case. This completes the proof.

All cases have been covered. □

9 Final remarks

Understanding the automorphism group of a combinatorial object is certainly interesting on its own. It could be interesting to also investigate the automorphism groups of generalized cluster complexes beyond finite type.

Another open problem is the following. As briefly mentioned in the introduction, the generalized cluster complex has a natural representation-theoretic interpretation (via an orbit category in the derived category of certain algebras, see [4] for a survey). In this context, $\mathcal{R}$ naturally identifies with a *shift functor*. It would be very interesting to also give a representation-theoretic interpretation of the involutive automorphisms in this context.

Acknowledgments

This work is partially supported by the ANR Combiné (Agence Nationale de la Recherche, ANR-19-CE48-0011). We thank the reviewer for their valuable comments and corrections that helped to improve this article.

References

[1] Ibrahim Assem, Ralf Schiffler, Vasilisa Shramchenko, *Cluster automorphisms*. Proc. Lond. Math. Soc. (3) 104(6) (2012) 1271–1302.

[2] Ibrahim Assem, Ralf Schiffler, and Vasilisa Shramchenko, *Cluster automorphisms and compatibility of cluster variables*. Glasg. Math. J. 56(3) (2014) 705–720.

[3] Tom Brady and Colum Watt, *Non-crossing partition lattices in finite real reflection groups*. Trans. Amer. Math. Soc. 360(4) (2008) 1983–2005.

[4] Aslak Bakke Buan, *An introduction to higher cluster categories.* Bull. Iran. Math. Soc. 37(2) (2011) 137–157.

[5] Wen Chang and Bin Zhu, *Cluster automorphism groups of cluster algebras of finite type*. J. Algebra 447 (2016) 490–515.

[6] Theo Douvropoulos and Matthieu Josuat-Vergès, *The generalized cluster complex: refined enumeration of faces and related parking spaces*. Symmetry, Integrability and Geometry: Methods and Applications (SIGMA) 19 (2023), 069, 40 pages.

[7] Sergey Fomin and Nathan Reading, *Generalized cluster complexes and Coxeter combinatorics.* Int. Math. Res. Notices 2005(44) (2005) 2709–2757.

[8] Sergey Fomin and Andrei Zelevinsky, *Y-systems and generalized associahedra*. Ann. Math. (2) 158(3) (2003) 977–1018.

[9] Siyang Liu and Fang Li, *Periodicities in cluster algebras and cluster automorphism groups*. Algebra Colloq. 28(4) (2021) 601–624.

[10] Christian Stump, Hugh Thomas, and Nathan Williams, *Cataland: why the Fuß?* Discrete Math. Theor. Comput. Sci., Proc. (2020) 1123–1134. Preprint, `https://arxiv.org/abs/1503.00710`.

[11] Hugh Thomas, *Defining an m-cluster category*. J. Algebra 318(1) (2007) 37–46.

[12] Eleni Tzanaki, *Faces of generalized cluster complexes and noncrossing partitions.* SIAM J. on Disc. Math. 22 (2008) 15–30.

[13] Bin Zhu, *Generalized cluster complexes via quiver representations*. J. Algebr. Comb. 27(1) (2008) 35–54.

Contact Information

Matthieu Josuat-Vergès
matthieu.josuat-verges@irif.fr

Centre National de la Recherche Scientifique (CNRS)
Institut de Recherche en Informatique Fondamentale (IRIF)
Université Paris-Cité, Paris, FRANCE.
https://orcid.org/0000-0002-7782-2171