**American Journal of Combinatorics** Research Article
Volume 4 (2025), Pages 79–92

# The $\mathscr{G}$-average of adjacency and Laplacian polynomials of a graph

Yaoping Hou* and Wenjun Xie

(Communicated by Sudipta Mallik)

**Abstract**

Let $\mathscr{G}$ be a finite multiplicative group of quaternion unit $U(\mathbb{H})$. The $\mathscr{G}$-average of adjacency (resp., Laplacian) polynomial of a graph $G$ is defined as the arithmetic mean of the characteristic polynomials of adjacency (resp., Laplacian) matrice of all $\mathscr{G}$-gain graphs on $G$. In this paper, we prove that the $\mathscr{G}$-average adjacency (resp., Laplacian) polynomial of a graph for any non-trivial finite subgroup $\mathscr{G}$ of $U(\mathbb{H})$ coincides with its matching (resp., weighted TU-subgraph) polynomial, which generalizes previous findings for signed graphs.

## 1 Introduction

Throughout this paper, all graphs are undirected and simple unless otherwise specified. Let $G$ be a graph with vertex set $V(G) = \{v_1, v_2, \cdots, v_n\}$ and edge set $E(G) \subseteq \binom{V(G)}{2}$. Various graph polynomials have been extensively studied, such as the matching polynomial and weighted TU-subgraph polynomial. The matching polynomial of a graph $G$, first introduced in [7], is defined as

$$M(G, x) = \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^t \Phi_t(G) x^{n-2t},$$

where

$$\Phi_t(G) = \left| \left\{ \mathscr{M}_t(G) : \mathscr{M}_t(G) \text{ is a matching of } G \text{ with } t \text{ edges, } 0 \le t \le \left\lfloor \frac{n}{2} \right\rfloor \right\} \right|.$$

* Corresponding author
MSC2020: 05C50, 05C22, 05C35; Keywords: $\mathscr{G}$-average of adjacency polynomial, $\mathscr{G}$-average of Laplacian polynomial, Quaternion unit gain graph, Matching polynomial, Weighted TU-subgraph polynomial
Received Apr 7, 2025; Revised Sep 10, 2025; Accepted Dec 22, 2025; Published Dec 31, 2025

Furthermore, a spanning subgraph $H$ of $G$ is called a TU-subgraph, as defined in [17], if every component of $H$ is either acyclic or unicyclic. Note that the concept of a TU-subgraph in this paper differs from that in [2], whose components are restricted to trees or odd unicyclic graphs. The weight of the TU-subgraph $H$ of $G$ is $w(H) = 2^c \prod_{i=1}^{t} |V(T_i)|$ if $H$ is an union of disjoint $t$ tree components $T_i$ and $c$ unicyclic components $U_j$ of $G$. In particular, $w(H) = 2^c$ if $H$ contains no tree components. Accordingly, the weighted TU-subgraph polynomial of $G$ is defined as

$$W(G,x) = \sum_{k=0}^{n} (-1)^k \sum_{H \in \mathscr{H}_k} w(H) x^{n-k},$$

where

$$\mathscr{H}_k = \{H : H \text{ is a TU-subgraph of } G \text{ and } |E(H)| = k\}.$$

Godsil and Gutman [6] (resp., Zhang and Chen [17]) have shown that the matching polynomial (resp., the weighted TU-subgraph polynomial) of a graph $G$ is exactly the average of adjacency (resp. Laplacian) polynomials of all signed graphs on $G$. Noting that every signed graph is a special gain graph, now we introduce the definition of gain graphs. Let

$$\mathbb{H} = \{h_1 + h_2\mathbf{i} + h_3\mathbf{j} + h_4\mathbf{k} : h_1, h_2, h_3, h_4 \in \mathbb{R}, \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}$$

denote the quaternion algebra equipped with a norm $|\cdot|$ such that $|h| = \sqrt{h_1^2 + h_2^2 + h_3^2 + h_4^2}$ for every quaternion $h = h_1 + h_2\mathbf{i} + h_3\mathbf{j} + h_4\mathbf{k}$. It can be verified that $U(\mathbb{H}) = \{h \in \mathbb{H} : |h| = 1\}$ forms a multiplicative group, i.e., the quaternion unit group. Let $\Gamma$ be a multiplicative group of quaternion unit. A graph $G = (V(G), E(G))$ may be viewed as a directed graph $(V(G), \overleftrightarrow{E(G)})$ by thinking of every edge as a pair of oppositely directed arcs. A directed weighted graph $G_\xi = (G, \xi)$ is called $\Gamma$-gain graph on a graph $G$ in [3] if the weight function $\xi : \overleftrightarrow{E(G)} \to \Gamma$ satisfies $\xi((v_j, v_i))\xi((v_i, v_j)) = 1$ for all $(v_i, v_j) \in \overleftrightarrow{E(G)}$. The function $\xi$ is also referred to as a gain function while $\Gamma$ is called the gain group of $G_\xi$. The images of the gain function $\xi$ is denoted by $\mathrm{Im}\xi$. In particular, an ordinary graph $G$ could be viewed as the $\Gamma$-gain graph $G_1$, where $\mathrm{Im}1 = \{1\}$. A $\Gamma$-gain graph $G_\xi$ is signed graph (resp., complex unit $\mathbb{T}$-gain graph, quaternion unit $U(\mathbb{H})$-gain graph) if and only if $\mathrm{Im}\xi \subseteq \{-1, 1\}$ (resp., $\mathrm{Im}\xi \subseteq \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathrm{Im}\xi \subseteq U(\mathbb{H})$). Therefore all of ordinary graphs, signed graphs and $\mathbb{T}$-gain graphs are special $U(\mathbb{H})$-gain graphs.

There are many classical results on theory of graph spectra and applications [4]. There are various generalization on spectra of gain graphs, such as spectra of signed graphs [8, 9], $\mathbb{T}$-gain graph [11, 13]. Recently, Belardo, Brunetti, Coble, Reff and Skogman extended spectral theory of $\mathbb{T}$-gain graph to $U(\mathbb{H})$-gain graph [3].

The adjacency (resp., Laplacian) polynomial is the characteristic polynomial of an adjacency (resp., Laplacian) matrix of a $U(\mathbb{H})$-gain graph, and the coefficient theorems with respect to them have been established in [3].

**Definition 1.1.** Let $\mathscr{G}$ be a finite subgroup of $U(\mathbb{H})$, $G$ be a graph with $m$ edges and $\Sigma(G, \mathscr{G})$ be the set consisting of all gain functions on $\overleftrightarrow{E(G)}$. Suppose that $P(G_\xi, x)$ (resp., $Q(G_\xi, x)$)

is the adjacency (resp., Laplacian) polynomial of a $\mathscr{G}$-gain graph $G_\xi$ for some $\xi \in \Sigma(G, \mathscr{G})$. Then

$$\bar{P}_{\mathscr{G}}(G,x) = \frac{1}{|\mathscr{G}|^m} \sum_{\xi \in \Sigma(G,\mathscr{G})} P(G_\xi, x) \text{ and } \bar{Q}_{\mathscr{G}}(G,x) = \frac{1}{|\mathscr{G}|^m} \sum_{\xi \in \Sigma(G,\mathscr{G})} Q(G_\xi, x)$$

are called the $\mathscr{G}$-average of adjacency polynomials of $G$ and $\mathscr{G}$-average of Laplacian polynomials of $G$ respectively.

Recall the relationship between the matching polynomial (resp., the weighted TU-subgraph polynomial) of $G$ and the adjacency (resp., Laplacian) polynomials of all signed graphs on $G$. With the notion in Definition 1.1, Godsil and Gutman proved that

$$\bar{P}_{\{-1,1\}}(G,x) = M(G,x) \tag{1.1}$$

in [6, Corollary 2.2] and Zhang and Chen proved that

$$\bar{Q}_{\{-1,1\}}(G,x) = W(G,x) \tag{1.2}$$

in [17, Theorem 2.2].

The aim of this paper is to generalize Eqs. (1.1) and (1.2) for gain graphs. Observing the gain group of signed graphs is $\{-1, 1\}$, it is natural to ask whether the Eq. (1.1) and (1.2) hold whenever $\{-1, 1\}$ is replaced by any non-trivial finite subgroup $\mathscr{G}$ of $U(\mathbb{H})$. In Section 2, we recall basic results on coefficient theorems with respect to the adjacency and Laplacian polynomials of the $U(\mathbb{H})$-gain graph and the classification of all finite subgroups of $U(\mathbb{H})$. In Section 3, we will prove that Eq. (1.1) and (1.2) hold in the case of any non-trivial finite subgroup of quaternion units, i.e., the matching (resp., weighted TU-subgraph) polynomial of a graph is equal to the $\mathscr{G}$-average adjacency (resp., Laplacian) polynomial of this graph for any non-trivial finite subgroup $\mathscr{G}$ of $U(\mathbb{H})$.

# 2 Preliminaries

## 2.1 The basic results for $U(\mathbb{H})$-gain graphs

Let $G$ be a graph and gain function $\xi \in \Sigma(G, U(\mathbb{H}))$. The adjacency matrix $M_A(G_\xi) = (a_{ij}) \in (U(\mathbb{H}) \cup \{0\})^{n \times n}$ of $G_\xi$ is defined by

$$a_{ij} = \begin{cases} \xi((v_i, v_j)), & \text{if } (v_i, v_j) \in \overleftrightarrow{E(G)} \\ 0, & \text{otherwise} \end{cases},$$

and the Laplacian matrix $M_L(G_\xi)$ of $G_\xi$ is defined to be $M_L(G_\xi) = M_D(G) - M_A(G_\xi)$, where $M_D(G) = \text{diag}(d(v_i) : v_i \in V(G))$ is the degree matrix of $G$ and $d(v_i)$ is the degree of vertex $v_i$. It is clear that $M_A(G_\xi)$ and $M_L(G_\xi)$ are both Hermitian for all $U(\mathbb{H})$-gain graph $G_\xi$.

The Moore determinant of a Hermitian quaternion matrix is defined in [5, 12], written as $\text{Mdet}(H)$ for the Hermitian quaternion matrix $H$. Moreover,

$$P(G_\xi, x) = \text{Mdet}(xI_n - M_A(G_\xi)) \text{ and } Q(G_\xi, x) = \text{Mdet}(xI_n - M_L(G_\xi))$$

are defined as the adjacency polynomial of $G_\xi$ and the Laplacian polynomial of $G_\xi$ respectively in [3].

Let $\xi(W) = \xi((v_1, v_2))\xi((v_2, v_3)) \cdots \xi((v_{s-1}, v_s))$ be the gain of a walk $W = v_1v_2v_3 \cdots v_s$ in $G_\xi$. A $U(\mathbb{H})$-gain graph $G_\xi$ is balanced if $\xi(C) = 1$ for every cycle $C$ in $G_\xi$. Two $U(\mathbb{H})$-gain graphs $G_{\xi_1}$ and $G_{\xi_2}$ are said to be switching equivalent, written as $G_{\xi_1} \sim G_{\xi_2}$, if there exists an invertible diagonal matrix $D \in (U(\mathbb{H}) \cup \{0\})^{n\times n}$ such that $D^*M_A(G_{\xi_1})D = M_A(G_{\xi_2})$, where $D^*$ is the conjugate transpose of $D$. It is obvious that this diagonal matrix $D$ is unitary, and then two switching equivalent $U(\mathbb{H})$-gain graphs have the same adjacency (resp., Laplacian) polynomial. The function $\zeta(v_i) = D_{ii}, i = 1, 2, \cdots, |V(G)|$ is called switching function and the gain function $\xi_2$ could be written as $\xi_1^\zeta$.

**Lemma 2.1.** *[3, Lemma 4.1 and 6.1] Two switching equivalent $U(\mathbb{H})$-gain graphs have the same adjacency (resp., Laplacian) polynomial and adjacency (resp., Laplacian) spectrum.*

The next result answers what are the most simple $U(\mathbb{H})$-gain graphs up to switching equivalence.

**Lemma 2.2.** *[16, The proof of Lemma 5.3] Let $G$ be a graph and $F$ be a spanning forest of $G$. Then for any gain function $\xi \in \Sigma(G, U(\mathbb{H}))$, there exists a $U(\mathbb{H})$-gain graph $G_\chi$ such that $\chi(\overrightarrow{e}) = 1$ for any $\overrightarrow{e} \in \overleftrightarrow{E(F)}$ and $G_\chi \sim G_\xi$.*

The following lemma is immediate.

**Lemma 2.3.** *[3, Lemma 2.20] Let $G_\xi$ be a $U(\mathbb{H})$-gain graph. Then $G_\xi$ is balanced if and only if $G_\xi \sim G_1$.*

**Lemma 2.4.** *[3, Lemma 2.19] Let $W$ be a closed walk in $U(\mathbb{H})$-gain graph $G_\xi$ with initial vertex $x$ and $\zeta : V(G) \to U(\mathbb{H})$ be a switching function. Then $\xi^\zeta(W) = \zeta(x)^{-1}\xi(W)\zeta(x)$, that is, $\xi^\zeta(W)$ and $\xi(W)$ are similar in $\mathbb{H}$.*

The Lemma 2.4 shows that the gain of a closed walk is an invariant of switching equivalence up to similarity. The real part and the imaginary part of a quaternion $h = h_1 + h_2\mathbf{i} + h_3\mathbf{j} + h_4\mathbf{k}$ are $h_1$ and $h_2\mathbf{i} + h_3\mathbf{j} + h_4\mathbf{k}$ in turn. It was shown that two quaternions are similar if and only if they have the same real part and modulus of the imaginary part respectively, see [3, Lemma 2.2]. Hence, the real part of the gain of a closed walk is exactly an invariant of switching equivalence.

A subgraph $B$ of $G$ is called basic if all components of $B$ are either edges or cycles. Let $H$ be a TU-subgraph of $G$ which is an union of disjoint $t$ acyclic components $T_i$ and $c$ unicyclic components $U_j$ with unique cycle $C_j$. Then the weight of $H$ under the gain function $\xi \in \Sigma(G, U(\mathbb{H}))$ is defined as

$$w_\xi(H) = \prod_{i=1}^{t} |V(T_i)| \prod_{j=1}^{c} (2 - 2\mathrm{Re}(\xi(C_j))).$$

Hence $w_\xi(H) > 0$ if and only if all $C_j$ are unbalanced under $\xi$.

**Lemma 2.5.** *[3, Theorem 4.6] (Coefficient Theorem with respect to adjacency polynomial of quaternion unit gain graph). Let $G_\xi$ be a $U(\mathbb{H})$-gain graph with order $n$ and size $m$. Suppose the adjacency polynomial of $G_\xi$ is $P(G_\xi, x) = x^n + \sum_{i=1}^{n} a_i(\xi)x^{n-i}$. Then*

$$a_i(\xi) = \sum_{B\in\mathscr{B}_i(G)} (-1)^{p(B)} 2^{c(B)} \prod_{C\in\mathscr{C}(B)} \mathrm{Re}(\xi(C)),$$

*where $\mathscr{B}_i(G) = \{B_i(G) : B_i(G)$ is a basic subgraph of $G$ and $|V(B_i(G))| = i\}$, $i = 1, 2, \cdots, n$, $p(B) = \#\{$the component of $B\}$, and $\mathscr{C}(B) = \{C_B : C_B$ is a cycle in $B\}$ with cardinal $c(B)$.*

**Lemma 2.6.** *[3, Theorem 6.9] (Coefficient Theorem with respect to Laplacian polynomial of quaternion unit gain graph). Let $G_\xi$ be a $U(\mathbb{H})$-gain graph with order $n$ and size $m$. Suppose the Laplacian polynomial of $G_\xi$ is $Q(G_\xi, x) = \sum_{k=0}^{n} (-1)^k b_k(\xi)x^{n-k}$. Then*

$$b_k(\xi) = \sum_{H\in\mathscr{H}_k} w_\xi(H).$$

Noting that the Moore determinant $\mathrm{Mdet}(A) = \det(A)$ when $A \in \mathbb{C}^{n\times n}$, Lemma 2.5 is the generalization of [11, Corollary 3.1] and Lemma 2.6 is the extension of [2, Theorem 3.9].

## 2.2 All finite subgroups of $U(\mathbb{H})$

Suppose that $\mathscr{G}$ is a finite multiplicative subgroup of $\mathbb{H}$. For every $q \in \mathscr{G}$, we have $q^{|\mathscr{G}|} = 1$ and then $|q| = 1$, i.e., $q \in U(\mathbb{H})$. Hence, $\mathscr{G}$ is a finite subgroup of $U(\mathbb{H})$ if and only if $\mathscr{G}$ is a finite multiplicative subgroup of $\mathbb{H}$. All kinds of binary polyhedral groups are listed in [15], which are:

(i) binary dihedral group (order $4\ell$) defined as $\mathscr{D}_{4\ell} = \langle a, b : a^{2\ell} = b^4 = 1, bab^{-1} = a^{-1}\rangle$,

(ii) binary tetrahedral group (order 24) defined as $\mathscr{T} = \langle a, b : a^6 = 1, b^3 = a^3, bab^{-1} = a^{-1}b\rangle$,

(iii) binary octahedral group (order 48) defined as $\mathscr{O} = \langle a, b : a^8 = 1, b^3 = a^4, bab^{-1} = a^{-1}b\rangle$,

(iv) and binary icosahedral group (order 120) defined as $\mathscr{I} = \langle a, b : a^{10} = 1, b^3 = a^5, bab^{-1} = a^{-1}b\rangle$.

Indeed, Amitsur classified all finite multiplicative subgroups of $\mathbb{H}$, consisting of cyclic groups and binary polyhedral groups listed above [1].

**Lemma 2.7.** *[1, Theorem 11] The finite multiplicative subgroups of $\mathbb{H}$ are the cyclic group of any order, the binary dihedral group of order $4\ell$, the groups $\mathscr{T}$, $\mathscr{O}$ and $\mathscr{I}$.*

For simplicity, we consider the following groups:

$$
\begin{gathered}
R_\ell = \{1, \omega_\ell, \omega_\ell^2, \cdots, \omega_\ell^{\ell-1}\}, \\
J_{4\ell} = \{1, \omega_{2\ell}, \omega_{2\ell}^2, \cdots, \omega_{2\ell}^{2\ell-1}, \mathbf{j}, \omega_{2\ell}\mathbf{j}, \omega_{2\ell}^2\mathbf{j}, \cdots, \omega_{2\ell}^{2\ell-1}\mathbf{j}\}, \\
Q_{24} = \left\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \tfrac{\pm 1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k}}{2}\right\}, \\
Q_{48} = \left\{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \tfrac{\pm 1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k}}{2}, \tfrac{\pm 1\pm\mathbf{i}}{\sqrt{2}}, \tfrac{\pm 1\pm\mathbf{j}}{\sqrt{2}}, \tfrac{\pm 1\pm\mathbf{k}}{\sqrt{2}}, \tfrac{\pm\mathbf{i}\pm\mathbf{j}}{\sqrt{2}}, \tfrac{\pm\mathbf{j}\pm\mathbf{k}}{\sqrt{2}}, \tfrac{\pm\mathbf{k}\pm\mathbf{i}}{\sqrt{2}}\right\}, \\
Q_{120} = \left\{\begin{gathered}
\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \tfrac{\pm 1\pm\mathbf{i}\pm\mathbf{j}\pm\mathbf{k}}{2}, \tfrac{\pm\mathbf{i}\pm e'\mathbf{j}\pm e\mathbf{k}}{2}, \tfrac{\pm e\mathbf{i}\pm\mathbf{j}\pm e'\mathbf{k}}{2}, \tfrac{\pm e'\mathbf{i}\pm e\mathbf{j}\pm\mathbf{k}}{2}, \\
\tfrac{\pm 1\pm e\mathbf{j}\pm e'\mathbf{k}}{2}, \tfrac{\pm 1\pm e'\mathbf{i}\pm e\mathbf{k}}{2}, \tfrac{\pm 1\pm e\mathbf{i}\pm e'\mathbf{j}}{2}, \tfrac{\pm e\pm e'\mathbf{j}\pm\mathbf{k}}{2}, \tfrac{\pm e\pm\mathbf{i}\pm e'\mathbf{k}}{2}, \tfrac{\pm e\pm e'\mathbf{i}\pm\mathbf{j}}{2}, \\
\tfrac{\pm e'\pm e\mathbf{j}\pm\mathbf{k}}{2}, \tfrac{\pm e'\pm e\mathbf{i}\pm\mathbf{k}}{2}, \tfrac{\pm e'\pm\mathbf{i}\pm e\mathbf{j}}{2}
\end{gathered}\right\},
\end{gathered}
$$

for $\omega_s = \mathrm{e}^{\frac{2\pi}{s}\mathbf{i}}$, $e = \frac{1+\sqrt{5}}{2}$ and $e' = \frac{1-\sqrt{5}}{2}$. In particular, $J_8 = \{\pm 1, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}\}$ is known as the quaternion group. It can be checked that all of homomorphisms

$$
\begin{gathered}
\sigma_1 : \mathscr{C}_\ell = \langle a : a^\ell = 1\rangle \to R_\ell, a \mapsto \omega_\ell, \\
\sigma_2 : \mathscr{D}_{4\ell} \to J_{4\ell}, a \mapsto \omega_{2\ell}, b \mapsto \mathbf{j}, \\
\sigma_3 : \mathscr{T} \to Q_{24}, a \mapsto \tfrac{1+\mathbf{i}+\mathbf{j}+\mathbf{k}}{2}, b \mapsto \tfrac{1+\mathbf{i}+\mathbf{j}-\mathbf{k}}{2}, \\
\sigma_4 : \mathscr{O} \to Q_{48}, a \mapsto \tfrac{1+\mathbf{i}}{\sqrt{2}}, b \mapsto \tfrac{1+\mathbf{i}+\mathbf{j}+\mathbf{k}}{2}, \\
\sigma_5 : \mathscr{I} \to Q_{120}, a \mapsto \tfrac{e+e'\mathbf{j}+\mathbf{k}}{2}, b \mapsto \tfrac{1-\mathbf{i}-\mathbf{j}+\mathbf{k}}{2}
\end{gathered}
$$

are isomorphisms. Therefore, every finite subgroup of $U(\mathbb{H})$ is one of $R_\ell$, $J_{4\ell}$, $Q_{24}$, $Q_{48}$ and $Q_{120}$ up to isomorphism.

Let $\mathscr{G}, \mathscr{H}$ be two finite subgroups of $U(\mathbb{H})$ and then $\mathscr{G}$ and $\mathscr{H}$ are said to be conjugate in $U(\mathbb{H})$ if there exists $q \in U(\mathbb{H})$ such that $\mathscr{G} = q^{-1}\mathscr{H}q$. It is evident that conjugacy of finite subgroups of $U(\mathbb{H})$ is an equivalence relation. Moreover, we have the following result:

**Lemma 2.8.** *[14, Theorem 3.1] Two finite subgroups of $U(\mathbb{H})$ are isomorphic if and only if they are conjugate in $U(\mathbb{H})$.*

Thus, it is not hard to calculate the sum of all elements of a non-trivial finite group of quaternion unit, shown in Proposition 2.9, which is an amazing conclusion and shows that all finite groups of quaternion unit have a perfect algebraic structure.

**Proposition 2.9.** The sum of all elements of a non-trivial finite subgroup of $U(\mathbb{H})$ is zero.

*Proof.* Firstly, we claim that $\sum\limits_{h\in\mathscr{Q}} h = 0$ if the non-trivial group $\mathscr{Q}$ is one of $R_\ell$, $J_{4\ell}$, $Q_{24}$, $Q_{48}$, $Q_{120}$. Indeed, it is immediate whenever $\mathscr{Q}$ is one of $Q_{24}$, $Q_{48}$ and $Q_{120}$, since $q \in \mathscr{Q}$ if and only if $-q \in \mathscr{Q}$. If $\mathscr{Q} = R_\ell$, then $\ell \geq 2$ and

$$\sum_{h\in\mathscr{Q}} h = \sum_{s=0}^{\ell-1} \omega_\ell^s = \frac{\omega_\ell^\ell - 1}{\omega_\ell - 1} = 0.$$

If $\mathscr{Q} = J_{4\ell}$, then

$$\sum_{h\in\mathscr{Q}} h = \sum_{s=0}^{2\ell-1} \omega_{2\ell}^s + \sum_{s=0}^{2\ell-1} \mathbf{j}\omega_{2\ell}^s = \sum_{s=0}^{2\ell-1} \omega_{2\ell}^s + \mathbf{j}\sum_{s=0}^{2\ell-1} \omega_{2\ell}^s = 0.$$

Secondly, let $\mathscr{G}$ be any non-trivial finite subgroup of $U(\mathbb{H})$. By Lemma 2.8, we write $\mathscr{G} = q^{-1}\mathscr{Q}q$ for an quaternion unit $q$ and a non-trivial finite group $\mathscr{Q}$ which we have discussed above. Hence, we obtain

$$\sum_{g\in\mathscr{G}} g = \sum_{g\in q^{-1}\mathscr{Q}q} g = \sum_{h\in\mathscr{Q}} q^{-1}hq = q^{-1}\left(\sum_{h\in\mathscr{Q}} h\right) q = 0. \qquad \square$$

Proposition 2.9 and the following multinomial theorem are the keys to prove the main theorem (Theorem 3.1) in this paper.

**Lemma 2.10.** *[10, Theorem 2.1] (Multinomial Theorem). Let $x_1, x_2, \cdots, x_n$ be real numbers and $k$ be any non-negative integer. Then*

$$\left(\sum_{i=1}^{n} x_i\right)^k = \sum_{(k_1,k_2,\cdots,k_n)\in S_n^k} \frac{k!}{k_1!k_2!\cdots k_n!}\prod_{i=1}^{n} x_i^{k_i},$$

*where $S_n^k = \{(k_1, k_2, \cdots, k_n) : k_1, k_2, \cdots, k_n \in \mathbb{Z}_{\geq 0}, k_1 + k_2 + \cdots + k_n = k\}$.*

# 3 The $\mathscr{G}$-average of adjacency (resp., Laplacian) polynomials of a graph

The main theorem of this section is Theorem 3.1, which describes the $\mathscr{G}$-average of adjacency (resp., Laplacian) polynomial of a graph $G$ for any non-trivial finite subgroup $\mathscr{G}$ of $U(\mathbb{H})$ is equal to the case for $\mathscr{G} = \{-1, 1\}$.

**Theorem 3.1.** *Let $\mathscr{G}$ be a non-trivial finite subgroup of $U(\mathbb{H})$ and $G$ be a graph with $n$ vertices and $m$ edges. Suppose that $M(G, x)$ and $W(G, x)$ are the matching polynomial and the weighted TU-subgraph polynomial of $G$ respectively. Then*

$$\bar{P}_{\mathscr{G}}(G, x) = M(G, x) \tag{3.1}$$

*and*

$$\bar{Q}_{\mathscr{G}}(G, x) = W(G, x), \tag{3.2}$$

*where $\bar{P}_{\mathscr{G}}(G, x)$ and $\bar{Q}_{\mathscr{G}}(G, x)$ are defined in Definition 1.1.*

*Proof.* (i) We write $\bar{P}_{\mathscr{G}}(G, x) = x^n + \sum_{i=1}^{n} \bar{a}_i x^{n-i}$. Using Lemma 2.5, we have

$$\begin{aligned}
\bar{a}_i &= \frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\Sigma(G,\mathscr{G})} a_i(\xi) = \frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\Sigma(G,\mathscr{G})}\sum_{B\in\mathscr{B}_i(G)} (-1)^{p(B)}2^{c(B)}\prod_{C\in\mathscr{C}(B)}\mathrm{Re}(\xi(C))\\
&= \sum_{B\in\mathscr{B}_i(G)} \frac{1}{|\mathscr{G}|^m}(-1)^{p(B)}2^{c(B)}\sum_{\xi\in\Sigma(G,\mathscr{G})}\prod_{C\in\mathscr{C}(B)}\mathrm{Re}(\xi(C))\\
&= \sum_{B\in\mathscr{B}_i(G)} \frac{1}{|\mathscr{G}|^m}(-1)^{p(B)}2^{c(B)}\sum_{\xi\in\Sigma(G,\mathscr{G})}\prod_{g\in\mathscr{G}}(\mathrm{Re}(g))^{d(g,B,\xi)},
\end{aligned}$$

where $d(g, B, \xi) = \#\{\text{cycle in } B \text{ with gain } g \text{ under the gain function } \xi\}$.

Let $X_{d_1 d_2 \cdots d_\ell}$ denote the event of a fixed basic subgraph $B$ of $G$ possessing exact $d_k$ cyclic components with gain $g_k$ for $\mathscr{G} = \{g_1, g_2, \cdots, g_\ell\}$. Then the probability

$$\begin{aligned}\mathbb{P}\left[X_{d_1 d_2 \cdots d_\ell}\right] &= \left(\frac{1}{|\mathscr{G}|}\right)^{c(B)} \binom{c(B)}{d_1} \binom{c(B) - d_1}{d_2} \cdots \binom{c(B) - d_1 - d_2 - \cdots - d_{\ell-1}}{d_\ell} \\ &= \frac{1}{|\mathscr{G}|^{c(B)}} \cdot \frac{c(B)!}{d_1! d_2! \cdots d_\ell!},\end{aligned}$$

with $0 \le d_1, d_2, \cdots, d_\ell \le c(B)$ and $d_1 + d_2 + \cdots + d_\ell = c(B)$, since every $U(\mathbb{H})$-gain cycle is switching equivalent to the gain graph on this cycle with at most one edge not having gain 1 by Lemma 2.2. By Proposition 2.9 and Lemma 2.10, for a fixed basic subgraph $B$, we have

$$\begin{aligned}&\frac{1}{\mathscr{G}^m}(-1)^{p(B)} 2^{c(B)} \sum_{\xi \in \Sigma(G, \mathscr{G})} \prod_{g \in \mathscr{G}} (\mathrm{Re}(g))^{d(g,B,\xi)} \\ =& \frac{1}{|\mathscr{G}|^m}(-1)^{p(B)} 2^{c(B)} \sum_{(d_1, d_2, \cdots, d_\ell) \in S_\ell^{c(B)}} |\mathscr{G}|^m \cdot \mathbb{P}\left[X_{d_1 d_2 \cdots d_\ell}\right] \prod_{k=1}^{\ell} (\mathrm{Re}(g_k))^{d_k} \\ =& \frac{1}{|\mathscr{G}|^m}(-1)^{p(B)} 2^{c(B)} \sum_{(d_1, d_2, \cdots, d_\ell) \in S_\ell^{c(B)}} \frac{|\mathscr{G}|^m}{|\mathscr{G}|^{c(B)}} \cdot \frac{c(B)!}{d_1! d_2! \cdots d_l!} \prod_{k=1}^{\ell} (\mathrm{Re}(g_k))^{d_k} \\ =& (-1)^{p(B)} \left(\frac{2}{|\mathscr{G}|}\right)^{c(B)} \left(\sum_{k=1}^{\ell} \mathrm{Re}(g_k)\right)^{c(B)} = (-1)^{p(B)} \left(\frac{2}{|\mathscr{G}|}\right)^{c(B)} \left(\mathrm{Re}\left(\sum_{k=1}^{\ell} g_k\right)\right)^{c(B)} \\ =& (-1)^{p(B)} \left(\frac{2}{|\mathscr{G}|}\right)^{c(B)} \left(\mathrm{Re}\left(\sum_{g \in \mathscr{G}} g\right)\right)^{c(B)} = (-1)^{p(B)} \left(\frac{2}{|\mathscr{G}|}\right)^{c(B)} \cdot 0^{c(B)} \\ =& \begin{cases} 0, & \text{if } c(B) > 0 \\ (-1)^{p(B)}, & \text{if } c(B) = 0 \end{cases} = \begin{cases} 0, & \text{if } B \text{ has at least one cyclic component} \\ (-1)^{\frac{1}{2}|V(B)|}, & \text{if } B \cong \frac{1}{2}|V(B)| K_2 \end{cases}.\end{aligned} \tag{3.3}$$

The last equality holds because $B$ is basic subgraph of $G$. If $i$ is odd, $B$ has at least one cyclic component for all $B \in \mathscr{B}_i(G)$, which implies that $\bar{a}_i = 0$. If $i$ is even, for all $B \in \mathscr{B}_i(G)$, the left hand of the Eq. (3.3) is non-zero if and only if $B \cong \frac{i}{2} K_2$ , which implies that

$$\bar{a}_i = \sum_{B \in \mathscr{B}_i(G), B \cong \frac{i}{2} K_2} (-1)^{\frac{i}{2}} = (-1)^{\frac{i}{2}} \Phi_{\frac{i}{2}}(G) = (-1)^k \Phi_k(G)$$

with $i = 2k$. By the definition of the matching polynomial, the Eq. (3.1) holds.

(ii) We write $\bar{Q}_{\mathscr{G}}(G,x)=\sum\limits_{k=0}^{n}(-1)^k\bar{b}_k x^{n-k}$. By Lemma 2.6, we have

$$\begin{aligned}\bar{b}_k &= \frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\Sigma(G,\mathscr{G})} b_k(\xi) = \frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\Sigma(G,\mathscr{G})}\sum_{H\in\mathscr{H}_k} w_\xi(H)\\ &= \sum_{H\in\mathscr{H}_k}\frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\Sigma(G,\mathscr{G})} w_\xi(H)\\ &= \sum_{H\in\mathscr{H}_k}\frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\Sigma(G,\mathscr{G})}\prod_{i=1}^{t}|V(T_i)|\prod_{j=1}^{c}\left(2-2\mathrm{Re}\left(\xi\left(C_j\right)\right)\right)\\ &= \sum_{H\in\mathscr{H}_k}\frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\Sigma(G,\mathscr{G})}\prod_{i=1}^{t}|V(T_i)|\prod_{g\in\mathscr{G}}\left(2-2\mathrm{Re}\left(g\right)\right)^{d(g,H,\xi)}.\end{aligned}$$

Let $Y_{d_1d_2\cdots d_\ell}$ be the event of a fixed TU-subgraph $H$ of $G$ with exactly $d_k$ unicyclic components with gain $g_k$ for $\mathscr{G}=\{g_1,g_2,\cdots,g_\ell\}$ in $G_\xi$. Then the probability

$$\mathbb{P}\left[Y_{d_1d_2\cdots d_\ell}\right]=\left(\frac{1}{|\mathscr{G}|}\right)^c\binom{c}{d_1}\binom{c-d_1}{d_2}\cdots\binom{c-d_1-d_2-\cdots-d_{\ell-1}}{d_\ell}=\frac{1}{|\mathscr{G}|^c}\cdot\frac{c!}{d_1!d_2!\cdots d_\ell!},$$

with $0\le d_1,d_2,\cdots,d_\ell\le c$ and $d_1+d_2+\cdots+d_\ell=c$. By Proposition 2.9 and Lemma 2.10, for a fixed TU-subgraph $U$, we have

$$\begin{aligned}&\frac{1}{|\mathscr{G}|^m}\sum_{\xi\in\mathscr{U}(H,G,\mathscr{G})}\prod_{i=1}^{t}|V(T_i)|\prod_{g\in\mathscr{G}}\left(2-2\mathrm{Re}\left(g\right)\right)^{d(g,H,\xi)}\\ =&\frac{1}{|\mathscr{G}|^m}\prod_{i=1}^{t}|V(T_i)|\sum_{(d_1,d_2,\cdots,d_\ell)\in S_l^c}|\mathscr{G}|^m\cdot\mathbb{P}\left[Y_{d_1d_2\cdots d_\ell}\right]\prod_{k=1}^{\ell}\left(2-2\mathrm{Re}\left(g_k\right)\right)^{d_k}\\ =&\frac{1}{|\mathscr{G}|^m}\prod_{i=1}^{t}|V(T_i)|\sum_{(d_1,d_2,\cdots,d_\ell)\in S_l^c}\frac{|\mathscr{G}|^m}{|\mathscr{G}|^c}\cdot\frac{c!}{d_1!d_2!\cdots d_\ell!}\prod_{k=1}^{\ell}\left(2-2\mathrm{Re}\left(g_k\right)\right)^{d_k}\\ =&\frac{1}{|\mathscr{G}|^c}\prod_{i=1}^{t}|V(T_i)|\left(\sum_{k=1}^{\ell}\left(2-2\mathrm{Re}\left(g_k\right)\right)\right)^c=\frac{2^c}{|\mathscr{G}|^c}\cdot\prod_{i=1}^{t}|V(T_i)|\left(\ell-\sum_{k=1}^{\ell}\mathrm{Re}\left(g_k\right)\right)^c\\ =&\frac{1}{|\mathscr{G}|^c}\cdot w(H)\left(|\mathscr{G}|-\sum_{g\in\mathscr{G}}\mathrm{Re}\left(g\right)\right)^c=\frac{1}{|\mathscr{G}|^c}\cdot w(H)\left(|\mathscr{G}|-\mathrm{Re}\left(\sum_{g\in\mathscr{G}}g\right)\right)^c\\ =&\frac{1}{|\mathscr{G}|^c}\cdot w(H)\cdot|\mathscr{G}|^c=w(H).\end{aligned}$$

Therefore we obtain $\bar{b}_k=\sum\limits_{H\in\mathscr{H}_k}w(H)$, which implies the Eq. (3.2) holds. □

**Example 3.2.** Let $K_3$ be a triangle with vertex set $V(K_3)=\{v_1,v_2,v_3\}$ and $J_8$ be the quaternion group. All TU-subgraphs of $K_3$ are following: three TU-subgraphs isomorphic to an edge $K_2$, three TU-subgraphs isomorphic to a path $P_3$ and the unique TU-subgraph isomorphic to $K_3$. Moreover, $K_3$ has three matchings containing an edge and no other matching. Therefore, we have

$$M(K_3, x) = x^3 - 3x \text{ and } W(K_3, x) = x^3 - 6x^2 + 9x - 2.$$

On the other hand, we consider $\xi_1, \xi_2, \cdots, \xi_8 \in \Sigma(K_3, J_8)$, which satisfy

$$\begin{gathered}
\xi_1((v_1, v_2)) = \xi_1((v_2, v_3)) = \xi_1((v_3, v_1)) = 1; \\
\xi_2((v_1, v_2)) = \xi_2((v_2, v_3)) = 1, \xi_2((v_3, v_1)) = -1; \\
\xi_3((v_1, v_2)) = \xi_3((v_2, v_3)) = 1, \xi_3((v_3, v_1)) = \mathbf{i}; \\
\xi_4((v_1, v_2)) = \xi_4((v_2, v_3)) = 1, \xi_4((v_3, v_1)) = \mathbf{-i}; \\
\xi_5((v_1, v_2)) = \xi_5((v_2, v_3)) = 1, \xi_5((v_3, v_1)) = \mathbf{j}; \\
\xi_6((v_1, v_2)) = \xi_6((v_2, v_3)) = 1, \xi_6((v_3, v_1)) = \mathbf{-j}; \\
\xi_7((v_1, v_2)) = \xi_7((v_2, v_3)) = 1, \xi_7((v_3, v_1)) = \mathbf{k}; \\
\xi_8((v_1, v_2)) = \xi_8((v_2, v_3)) = 1, \xi_8((v_3, v_1)) = \mathbf{-k};
\end{gathered}$$

For all $\xi \in \Sigma(K_3, J_8)$, there exists a graph $(K_3)_{\xi_j}$ possessing the same adjacency (resp., Laplacian) polynomial as those of $(K_3)_\xi$, $j = 1, 2, \cdots, 8$ by Lemma 2.1 and Lemma 2.2. According to Lemma 2.5 and 2.6, it is not hard to show that

$$\begin{gathered}
P((K_3)_{\xi_1}, x) = x^3 - 3x - 2, P((K_3)_{\xi_2}, x) = x^3 - 3x + 2, \\
P((K_3)_{\xi_j}, x) = x^3 - 3x, j = 3, 4, \cdots, 8; \\
Q((K_3)_{\xi_1}, x) = x^3 - 6x^2 + 9x, Q((K_3)_{\xi_2}, x) = x^3 - 6x^2 + 9x - 4, \\
Q((K_3)_{\xi_j}, x) = x^3 - 6x^2 + 9x - 2, j = 3, 4, \cdots, 8.
\end{gathered}$$

Consider $\Phi_i = \{\tau \in \Sigma(K_3, J_8) : (K_3)_\tau \sim (K_3)_{\xi_i}\}$ and

$$\begin{gathered}
\sigma_{ij} : \Phi_i \to \Phi_j, \tau_1 \mapsto \tau_2 \\
(\tau_2((v_1, v_2)) = \tau_1((v_1, v_2)), \tau_2((v_2, v_3)) = \tau_1((v_2, v_3)), \tau_2((v_3, v_1)) = \tau_1((v_3, v_1))x^{-1}y),
\end{gathered}$$

for $\tau_1((v_1, v_2))\tau_1((v_2, v_3))\tau_1((v_3, v_1)) = x$, $\tau_2((v_1, v_2))\tau_2((v_2, v_3))\tau_2((v_3, v_1)) = y$ and $i, j = 1, 2, \cdots, 8$. It is clear that every $\sigma_{ij}$ is a bijection. Hence, there are exactly $8^2$ distinct $J_8$-gain graphs $(K_3)_\xi$ share the same adjacency (resp., Laplacian) polynomial with $(K_3)_{\xi_j}$ for all $j = 1, 2, \cdots, 8$. Therefore, we obtain

$$\bar{P}_{J_8}(K_3, x) = \frac{8^2}{8^3} \sum_{j=1}^{8} P((K_3)_{\xi_j}, x) = x^3 - 3x = M(K_3, x),$$

$$\bar{Q}_{J_8}(K_3, x) = \frac{8^2}{8^3} \sum_{j=1}^{8} Q((K_3)_{\xi_j}, x) = x^3 - 6x^2 + 9x - 2 = W(K_3, x).$$

The following result describes the property of forests, which is the only kind of graphs such that its adjacency (resp., Laplacian) polynomial is exactly equal to its $\mathscr{G}$-average of adjacency (resp., Laplacian) polynomials. It also shows that the condition "$\mathscr{G}$ is non-trivial" in Theorem 3.1 cannot be removed.

**Corollary 3.3.** *Let $\mathscr{G}$ be a finite subgroup of $U(\mathbb{H})$ and $G$ be a graph. Suppose that $P(G_\xi, x)$ (resp., $Q(G_\xi, x)$) is the adjacency (resp., Laplacian) polynomial of a $\mathscr{G}$-gain graph $G_\xi$. $M(G, x), W(G, x)$ are the matching polynomial and the weighted TU-subgraph polynomial of $G$ respectively. The following five statements are equivalent:*

*(i) $G$ is a forest;*

*(ii) $P(G_\xi, x) = M(G, x)$, for all $\xi \in \Sigma(G, \mathscr{G})$;*

*(iii) $P(G_\xi, x) = M(G, x)$, for some $\xi \in \Sigma(G, \mathscr{G})$ such that $G_\xi$ is balanced;*

*(iv) $Q(G_\xi, x) = W(G, x)$;*

*(v) $Q(G_\xi, x) = W(G, x)$, for some $\xi \in \Sigma(G, \mathscr{G})$ such that $G_\xi$ is balanced.*

*Proof.* (i)⇒(ii) and (i)⇒(iv): It is immediate if $\mathscr{G} = \{1\}$ and now we suppose $\mathscr{G}$ is nontrivial. $G_\xi$ is balanced for all $\xi \in \Sigma(G, \mathscr{G})$ since the forest $G$ is acyclic. According to Lemma 2.1 and Lemma 2.3, $P(G_\xi, x) = P(G_1, x)$ and $Q(G_\xi, x) = Q(G_1, x)$ for all $\xi \in \Sigma(G, \mathscr{G})$. Furthermore, by Theorem 3.1 and Definition 1.1, we have

$$M(G, x) = \bar{P}_{\mathscr{G}}(G, x) = P(G_\xi, x) \text{ and } W(G, x) = \bar{Q}_{\mathscr{G}}(G, x) = Q(G_\xi, x).$$

(ii)⇒(iii) and (iv)⇒(v): Both are obvious.

(iii)⇒(i): For $\xi \in \Sigma(G, \mathscr{G})$ such that $G_\xi$ is balanced, $G_1 \sim G_\xi$, which implies that $P(G, x) = P(G_1, x) = P(G_\xi, x) = M(G, x)$. Since the matching polynomial of a graph coincides with its adjacency polynomial if and only if the graph is a forest [6, Corollary 2.1], we obtain $G$ is a forest.

(v)⇒(i): As the same way to the proof of (iii)⇒(i), we have $Q(G, x) = W(G, x) = \sum_{k=0}^{n} (-1)^k \sum_{H \in \mathscr{H}_k} w(H) x^{n-k}$. By Theorem 7.2.8 of [4], then

$$Q(G, x) = \sum_{k=0}^{n} (-1)^k \sum_{|E(F)|=k} \mathscr{P}(F) x^{n-k},$$

where the sum is taken over all spanning forests $F$, and $\mathscr{P}(F)$ is the product of the numbers of vertices in the components of $F$. Therefore,

$$\sum_{H \in \mathscr{H}_k} w(H) = \sum_{|E(F)|=k} \mathscr{P}(F)$$

for all $k = 0, 1, 2, \cdots, n$. Assume that $G$ has at least one cycle $C_t$. Then $C_t \cup (n-t)K_1$ is a TU-subgraph of $G$ with $t$ edges. Noting that all spanning forests $F$ with $t$ edges are also TU-subgraphs of $G$, we have

$$\sum_{H \in \mathscr{H}_t} w(H) \geq 2 + \sum_{|E(F)|=t} \mathscr{P}(F) > \sum_{|E(F)|=t} \mathscr{P}(F),$$

a contradiction. Hence, $G$ is a forest. □

Naturally, we wonder whether there exists a gain function $\xi \in \Sigma(G, U(\mathbb{H}))$ such that $P(G_\xi, x) = M(G, x)$ and $Q(G_\xi, x) = W(G, x)$ for $G$ containing some cycles. In Example

3.2, gain functions $\xi_j, j = 3, \cdots, 8$ satisfy this condition. If there exists a gain function $\xi \in \Sigma(G, U(\mathbb{H}))$ such that $\mathrm{Re}(\xi(C)) = 0$ for all cycles $C$ in $G$, then $P(G_\xi, x) = M(G, x)$ and $Q(G_\xi, x) = W(G, x)$ by Lemma 2.5 and Lemma 2.6. The following proposition shows that there is no such a gain function on the completed graph $K_4$.

**Proposition 3.4.** There is no gain function $\varphi \in \Sigma(K_4, U(\mathbb{H}))$ such that $\mathrm{Re}(\varphi(C)) = 0$ for all cycles $C$ in $K_4$.

*Proof.* Assume that there exists a gain function $\varphi \in \Sigma(K_4, U(\mathbb{H}))$ such that $\mathrm{Re}(\varphi(C)) = 0$ for all cycles $C$ in $K_4$ with vertex set $V(K_4) = \{v_1, v_2, v_3, v_4\}$. Define two switching function $\xi, \nu : V(K_4) \to U(\mathbb{H})$ as follows:

$$\xi(v_i) = \begin{cases} 1, \quad i = 1, 4, \\ (\varphi((v_1, v_2)))^{-1}, \quad i = 2, \\ (\varphi((v_1, v_2))\varphi((v_2, v_3)))^{-1}, \quad i = 3. \end{cases}, \nu(v_i) = \begin{cases} 1, \quad i = 1, \\ (\varphi^\xi((v_1, v_2)))^{-1}, \quad i = 2, \\ (\varphi^\xi((v_1, v_2))\varphi^\xi((v_2, v_3)))^{-1}, \quad i = 3, \\ (\varphi^\xi((v_1, v_2))\varphi^\xi((v_2, v_4)))^{-1}, \quad i = 4. \end{cases}.$$

It can be checked that $\varphi^{\xi\nu}((v_1, v_2)) = \varphi^{\xi\nu}((v_2, v_3)) = \varphi^{\xi\nu}((v_2, v_4)) = 1$. Recall the result of Lemma 2.4 implies the real part of the gain of a cycle is an invariant of switching equivalence. Without loss of generality, suppose that $\varphi((v_1, v_2)) = \varphi((v_2, v_3)) = \varphi((v_2, v_4)) = 1$.

We write $\varphi((v_3, v_4)) = b_1\mathbf{i} + c_1\mathbf{j} + d_1\mathbf{k}$, $\varphi((v_1, v_3)) = b_2\mathbf{i} + c_2\mathbf{j} + d_2\mathbf{k}$ and $\varphi((v_4, v_1)) = b_3\mathbf{i} + c_3\mathbf{j} + d_3\mathbf{k}$ since the real part of the gains of cycle $v_2v_3v_4v_2$, $v_1v_2v_3v_1$ and $v_1v_2v_4v_1$ are all zero, where $b_i^2 + c_i^2 + d_i^2 \neq 0$, $i = 1, 2, 3$. Now consider that the real part of any cycle containing edge $v_3v_4$ is zero, i.e.,

$$\mathrm{Re}(\varphi(v_1v_3v_4v_1)) = \mathrm{Re}(\varphi(v_1v_2v_4v_3v_1)) = \mathrm{Re}(\varphi(v_1v_4v_3v_2v_1)) = \mathrm{Re}(\varphi(v_1v_4v_2v_3v_1)) = 0.$$

Hence

$$(c_2d_3 - d_2c_3)b_1 + (d_2b_3 - b_2d_3)c_1 + (b_2c_3 - c_2b_3)d_1 = 0, \tag{3.4a}$$
$$b_1b_2 + c_1c_2 + d_1d_2 = 0, \tag{3.4b}$$
$$b_1b_3 + c_1c_3 + d_1d_3 = 0, \tag{3.4c}$$
$$b_2b_3 + c_2c_3 + d_2d_3 = 0. \tag{3.4d}$$

According to Eq. (3.4a), we obtain

$$\det \begin{pmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{pmatrix} = 0$$

and then the three vectors $(b_1, c_1, d_1), (b_2, c_2, d_2), (b_3, c_3, d_3)$ in $\mathbb{R}^3$ are coplanar. By Eq. (3.4b) and (3.4c), $(b_1, c_1, d_1)$ is orthogonal to both $(b_2, c_2, d_2)$ and $(b_3, c_3, d_3)$. Hence, $(b_2, c_2, d_2)$ is parallel to $(b_3, c_3, d_3)$ and we can write $(b_2, c_2, d_2) = t(b_3, c_3, d_3)$ for some $t \in \mathbb{R}$. Together with the Eq. (3.4d), $t(b_3^2 + c_3^2 + d_3^2) = 0$, which implies $b_2^2 + c_2^2 + d_2^2 = t^2(b_3^2 + c_3^2 + d_3^2) = 0$, a contradiction. □

## Acknowledgments

This research is supported by the National Natural Science Foundation of China (No. 12331012). The authors would like to thank the anonymous referees for their valuable comments and suggestions.

## References

[1] S. A. Amitsur, *Finite subgroup of division rings*, Transactions of American Mathematical Society 80 (1955) 361–386.

[2] F. Belardo, S. K. Simić, *On the Laplacian coefficients of signed graphs*, Linear Algebra Appl. 475 (2015) 94–113.

[3] F. Belardo, M. Brunetti, N. J. Coble, N. Reff, H. Skogman, *Spectra of quaternion unit gain graphs*, Linear Algebra Appl. 632 (2022) 15–49.

[4] D. Cvetković, P. Rowlinson, S. Simić, *An introduction to the theory of graph spectra*, Cambridge Univ. Press, Cambridge (2010).

[5] F. J. Dyson, *Quaternion determinants*, Helv. Phys. Acta 45 (1972) 289–302.

[6] C. D. Godsil, I. Gutman, *On the matching polynomial of a graph*, in: Algebraic Methods in Graph Theory, Vol. I, II (Szeged, 1978), in: Colloq. Math. Soc. János Bolyai, vol. 25, North-Holland, Amsterdam, New York (1981) 241–249.

[7] I. Gutman, *The acyclic polynomial of a graph*, Publ. Inst. Math., (Beograd) 22 (1977) 63–69.

[8] Y. Hou, J. Li, Y. Pan, *On the Laplacian eigenvalues of signed graphs*, Linear Multilinear Algebra 51 (1) (2003) 21–30.

[9] Y. Hou, *Bounds for the least Laplacian eigenvalue of a signed graph*, Acta Math. Sin. Engl. Ser. 21 (4) (2005) 955–960.

[10] K. K. Kataria, *A probabilistic proof of the multinomial theorem*, American Mathematical Monthly 123 (2016) 94–96.

[11] R. Mehatari, M. R. Kannan, A. Samanta, *On the adjacency matrix of a complex unit gain graph*, Linear Multilinear Algebra 70 (2022) 1798–1813.

[12] E. H. Moore, *On the determinant of an Hermitian matrix of quaternionic elements*, Bull. Am. Math. Soc. 28 (1922) 161–162.

[13] N. Reff, *Spectral properties of complex unit gain graphs*, Linear Algebra Appl. 436 (2012) 3165–3176.

[14] H. Takagi, *Conformally flat Riemannian manifolds admitting a transitive group of isometries II*, Tohoku Mathematical Joural 27 (1975) 445–451.

[15] G. Vincent, *Les groupes linéaires finis sans points fixés*, Comment. Math. Helv. vol. 20 (1947) 117–171.

[16] T. Zaslavsky, *Biased graphs. I. Bias, balance, and gains*, Journal of Combinatorial Theory, Series B 47 (1989) 32–52.

[17] Y. Zhang, H. Chen, *The average Laplacian polynomial of a graph*, Discrete Applied Mathematics 283 (2020) 737–743.

## Contact Information

Yaoping Hou
yphou@hunnu.edu.cn
College of Mathematics and Statistics, Hunan Normal University
Changsha, Hunan, 410081, China
https://orcid.org/0000-0001-8101-6968

Wenjun Xie
531865687@qq.com
College of Mathematics and Statistics, Hunan Normal University
Changsha, Hunan, 410081, China
https://orcid.org/0009-0006-5364-9008