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Computing the degree of some matchings in a graph

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Abstract

Let G be a connected graph. A matching M in G is a set of edges of G without two of them adjacent (having a common vertex). The graph whose vertices are the matchings in G and two matchings M and N are adjacent if and only if $(M \setminus N) \cup (N \setminus M)$ is the edge set of a path or a cycle, is denoted by $\mathcal{G}(\mathcal{M}(G))$ and called the skeleton of the matching polytope of G. The degree of a matching M in G is the degree of the vertex M in $\mathcal{G}(\mathcal{M}(G))$. In the literature some authors have studied the degree of some matchings, in particular when G is a tree. In this paper we continue this study and we present some formulas to compute the degree of a matching M in a graph with cycles. More explicitly, we focus on the matchings having 1 or 2 edges and on the matchings of more than two edges with some constraints.

1 Introduction

Let $G = (V(G), E(G))$ be a simple and connected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. For each k, with $1 \leq k \leq m$, the edge $e_k = \{v_i, v_j\}$, edge incident to vertices v_i and v_j of G, is simply denoted by v_iv_j. The degree of the vertex $v \in V(G)$ in the graph G is denoted by $d_G(v)$ and the set of neighbors of v in G (vertices adjacent to v) is denoted by $N_G(v)$. A path with $n \geq 2$ vertices, denoted by P_n, is a tree with $n-2$ vertices having degree 2 and the others having degree 1. A cycle with $n \geq 3$ vertices, denoted by C_n, is a connected graph with all vertices of degree 2. Thus, a cycle C_n is obtained from the path P_n by adding the edge incident to the vertices of degree 1 in P_n. We denote a path by its sequence of edges, $e_{i_1}, e_{i_2}, \ldots, e_{i_j}$, where e_{i_p} is adjacent to $e_{i_{p+1}}$, for $1 \leq p \leq j-1$, and a cycle obtained from this path is denoted by its edges, $e_{i_1}, e_{i_2}, \ldots, e_{i_j}, e_{i_{j+1}}$, where $e_{i_{j+1}}$ is adjacent to e_{i_1} and e_{i_j}. We say that P_n has length $n-1$ and C_n has length n. A vertex $v \in V(G)$ is a pendant vertex of G if $d_G(v) = 1$ and an edge $e \in E(G)$ is a pendant edge of G if it is incident to a pendant vertex of G. A matching in G is a subset $M \subseteq E(G)$ without two edges adjacent in G, that is, having a common vertex.

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Let M be a matching in G. A vertex v of G is M-saturated if there is an edge of M incident to v. Otherwise, v is an M-unsaturated vertex. As usual, $S(M)$ is the set of M-saturated vertices. The set $N_G(S(M))$ is the set $\bigcup_{v \in S(M)} N_G(v)$. A path P in G (respectively, a cycle C in G) where edges alternate between being in M and in $E(G) \setminus M$ is called an M-alternating path (respectively, cycle). Let M and N be two matchings in G, the symmetric difference of M and N is the set $M \Delta N = (M \setminus N) \cup (N \setminus M)$. Note that, if $M \Delta N$ is an M-alternating path (respectively, cycle), then it is also an N-alternating path (respectively, cycle). Moreover, if it is an M-alternating cycle, then the cycle has even length. For more basic definitions and notations of graphs, see [3, 5] and, of matchings, see [8].

Let us fix an order on the elements of $E(G)$ and let $\mathbb{R}^{E(G)}$ be the vector space of functions from $E(G)$ into $\mathbb{R}$. For $F \subseteq E(G)$ the incidence vector of F is

$$\chi_F(u) = \begin{cases} 1, & \text{if } u \in F \\ 0, & \text{otherwise.} \end{cases}$$

In general, we identify each subset of edges of G with its respective incidence vector. The matching polytope of G, $\mathcal{M}(G)$, is the convex hull of the incidence vectors of the matchings in G. For more definitions and notations of polytopes, see [7].

In [4], the authors studied the polytope $\mathcal{M}(G)$ when G is a tree. However, the definition of a graph obtained from this polytope, called the skeleton, and the notion of degree of a matching appeared in [1]. It is worth mentioning that this definition was connected only with trees. In [6] we have an algorithm to compute the degree of a matching when the graph is a tree. Lately, in [2], the authors started the generalization of the study of the degree of a vertex, that is a matching in a graph, in the mentioned graph called skeleton. In this paper, we continue this study.

The skeleton of $\mathcal{M}(G)$ is the graph $\mathcal{G}(\mathcal{M}(G))$ whose vertices and edges are, respectively, vertices and edges of $\mathcal{M}(G)$. Consequently, the vertex set of $\mathcal{G}(\mathcal{M}(G))$ is the set of matchings in G. The next result characterizes when two distinct matchings in G are adjacent in $\mathcal{G}(\mathcal{M}(G))$.

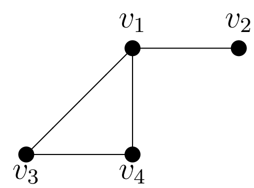
Theorem 1.1. *[9] Let G be a graph. Two distinct matchings M and N in G are adjacent in the matching polytope $\mathcal{M}(G)$ if and only if $M \Delta N$ is a path or a cycle in G.*

Note that the path (respectively, cycle) mentioned in Theorem 1.1 is an M-alternating path (respectively, cycle).

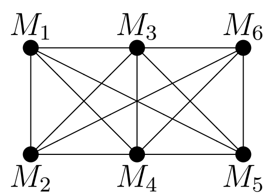
Using the last result we get that the empty matching in a graph is adjacent to matchings with a single edge. Moreover, if a matching has more than one edge, then it is not adjacent to the empty matching. Therefore, the degree of the empty matching in a graph is the number of edges of the graph. This result appears in [1] when the graph G is a tree and in [2] for any graph G.

In the next example we see the skeleton of the matching polytope of a graph with a cycle.

Example 1.2. Let G be the following graph.



The matchings in G are $M_1 = \emptyset$, $M_2 = \{v_1v_2\}$, $M_3 = \{v_1v_4\}$, $M_4 = \{v_1v_3\}$, $M_5 = \{v_3v_4\}$ and $M_6 = \{v_1v_2, v_3v_4\}$. So, the skeleton of $\mathcal{M}(G)$ is the graph $\mathcal{G}(\mathcal{M}(G))$.



Note that the matching M_2 is not adjacent to the matching M_5 because $M_2 \Delta M_5 = \{v_1v_2, v_3v_4\}$ is not a path nor a cycle.

We denote by $deg_G(M)$ the degree of the matching M in the graph $\mathcal{G}(\mathcal{M}(G))$. The focus of [1, 6] was the degree of a matching in the graph $\mathcal{G}(\mathcal{M}(T))$, where T was a tree. In [6], we have an algorithm to compute this degree. In [2], the authors characterized the matchings having minimum degree in $\mathcal{G}(\mathcal{M}(G))$, for any graph G. In this paper we obtain formulas to compute the degree of some matching in graphs having cycles.

In Section 2 we present some results obtained previously about the degree of a matching. In Section 3 we remove some vertices of the initial graph to divide the calculation of the degree of a matching M in a sum of degree of submatchings of M. In this section we introduce the operation of elimination on a graph with a matching. The Section 4 is dedicated to the degree of a matching with a single edge and in Section 5 we continue this study but in a matching with two edges. In Section 5 we describe another operation on the graph with a matching, the withdraw/choose. A formula to compute the degree of a matching with more than two edges and some constraints is the objective of Section 6. We conclude this paper with some final remarks and open questions in Section 7.

2 Preliminaries

Let G be a graph and M be a matching in G. We say that an M-alternating path, P, is an M-good path if each vertex of P of degree 1 in P is in $V(G) \setminus S(M)$ or the edge of P incident to it belongs to M.

Remark 2.1. Let M and N be two distinct matchings in G such that N is adjacent to M in $\mathcal{G}(\mathcal{M}(G))$. Then $M\Delta N$ is an M-good path or an M-alternating cycle. Consequently, all vertices of $M\Delta N$ belong to $N_G(S(M))$.

In [2], the M-good paths, P, are divided in three groups:

(1) P is an oo-M-path if its pendant edges belong to M.

(2) P is a cc-M-path if its pendant vertices are M-unsaturated.

(3) P is an oc-M-path if one of its pendant edges belongs to M and one of its pendant vertices is M-unsaturated.

In [1, 6, 2], the definition of M-good path and its subdivisions was important to prove which matchings are adjacent to a given matching, see Theorem 1.1. However, when the graph G has cycles, by Theorem 1.1, the symmetric difference of the two adjacent matchings can be a cycle. In this case, we obtain an M-alternating cycle with all vertices M-saturated. The next result shows us a way to obtain matchings not adjacent to a given matching.

Proposition 2.2. [2] Let M and N be matchings in a graph G. If M is adjacent to N in $\mathcal{G}(\mathcal{M}(G))$, then $||M| - |N|| \in \{0, 1\}$.

Throughout this paper, if the matching is a matching in a tree, we use the algorithm described in [6] to compute its degree.

3 Elimination

The first operation on G with respect to the matching M, called the elimination, consists of the elimination of the vertices of $V(G) \setminus N_G(S(M))$ from G. In some cases, with this operation on G we obtain subgraphs of G where the matching restriction of M to each one of these subgraphs has fewer edges than the matching M.

Proposition 3.1. Let G be a connected graph and M be a matching in G. Let $G^{(1)}, \ldots, G^{(r)}$ be the connected components of the subgraph of G induced by the vertex set $N_G(S(M))$ and $M^{(i)} = M \cap E(G^{(i)})$, for $1 \leq i \leq r$. Let $H = (V(H), E(H))$ be the subgraph of G induced by the vertex set $V(G) \setminus N_G(S(M))$. Then

$$deg_G(M) = \sum_{i=1}^{r} deg_{G^{(i)}}(M^{(i)}) + |E(H)|.$$

Proof. Let N be a matching in G adjacent to M in the graph $\mathcal{G}(\mathcal{M}(G))$. By Theorem 1.1 we conclude that $M \Delta N$ is a path or a cycle in G. Since $V(H) \cap N_G(S(M)) = \emptyset$ and N is a matching, by Remark 2.1 we have $|N \cap E(H)| = 0$ or $|N \cap E(H)| = 1$.

If $|N \cap E(H)| = 1$, then by Proposition 2.2 and Theorem 1.1 we get

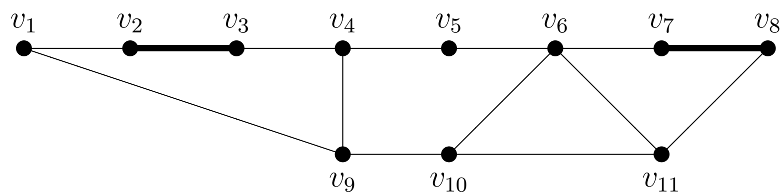
$$M = N \cap (E(G) \setminus E(H)).$$

Therefore, there are $|E(H)|$ matchings of these kind.

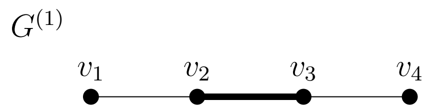
If $|N \cap E(H)| = 0$, then $N \subseteq \bigcup_{i=1}^{r} E(G^{(i)})$. Consequently, N is a matching in $\bigcup_{i=1}^{r} G^{(i)}$. Using Theorem 1.1 and the fact that N is different from M, we conclude that there is an i, with $1 \leq i \leq r$, such that $M \Delta N$ is a nontrivial $M^{(i)}$-good path or an M-alternating cycle in $G^{(i)}$. Moreover, if $j \neq i$ and $1 \leq j \leq r$, then $M^{(j)} \subset N$ and $N \cap E(G^{(i)})$ is a matching in $G^{(i)}$ adjacent to $M^{(i)}$.

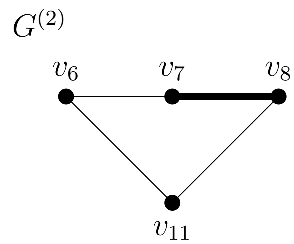
Conversely, for each edge $e \in E(H)$ we have the matching $M \cup \{e\}$ adjacent to M in $\mathcal{G}(\mathcal{M}(G))$. For each matching $R^{(i)}$ in $G^{(i)}$ adjacent to $M^{(i)}$ we have the matching $\bigcup_{j=1,\ j\neq i}^{r} M^{(j)} \cup R^{(i)}$ adjacent to M in $\mathcal{G}(\mathcal{M}(G))$. So, the result follows. □

Example 3.2. Let $M = \{v_2v_3, v_7v_8\}$ (edges of M in dark) the following matching in G:



As $N_G(S(M)) = \{v_1, v_2, v_3, v_4, v_6, v_7, v_8, v_{11}\}$, we get $V(G) \setminus N_G(S(M)) = \{v_9, v_{10}, v_5\}$. So, the subgraph H of G induced by the vertex set $V(G) \setminus N_G(S(M))$ has edge set $E(H) = \{v_9v_{10}\}$. Moreover, the connected components of the subgraph of G induced by the vertex set $N_G(S(M))$ are





So, by Proposition 3.1

$$\begin{aligned} deg_G(M) = &\ deg_{G^{(1)}}(M^{(1)}) + deg_{G^{(2)}}(M^{(2)}) + |E(H)| \\ = &\ deg_{G^{(1)}}(M^{(1)}) + deg_{G^{(2)}}(M^{(2)}) + 1, \end{aligned}$$

where $M^{(1)} = M \cap E(G^{(1)}) = \{v_2v_3\}$ and $M^{(2)} = M \cap E(G^{(2)}) = \{v_7v_8\}$.

As $G^{(1)}$ is a tree, using the algorithm described in [6] we get

$$deg_{G^{(1)}}(M^{(1)}) = 4.$$

So,

$$deg_G(M) = 4 + deg_{G^{(2)}}(M^{(2)}) + 1 = deg_{G^{(2)}}(M^{(2)}) + 5.$$

Note that, in the last example, we started with a matching having two edges and with the operation elimination on the graph, described in this section, we finished with a matching having only one edge (see Remark 2.1).

4 Matchings having only one edge

In [6] the formula to calculate the degree of a matching in a tree having only one edge use an operation called the duplication by the edge of the matching. If G has cycles and the unique edge of the matching belongs to one or more cycles, the duplication operation is not easy because we need to duplicate vertices and edges that do not belong to the matching. However, the next theorem shows the degree of this kind of matchings. This theorem is a particular case of Theorem 2.6 in [2]. The proof of this is different from that given in [2].

Theorem 4.1. *Let G be a connected graph and u and v be two vertices of G. Let $M = \{uv\}$ be a matching in G. Then*

$$deg_G(M) = d_G(u)d_G(v) - |N_G(u) \cap N_G(v)| + s,$$

where s is the number of edges in G not adjacent to uv.

Proof. We can assume that $V(G) \setminus N_G(S(M)) = \emptyset$. If not, first we use Proposition 3.1. Let N be a matching in G adjacent to M, in the matching polytope $\mathcal{M}(G)$. By Theorem 1.1 we conclude that $M\Delta N$ is an M-good path or an M-alternating cycle in G. Since $|M| = 1$, $M\Delta N$ is not a cycle. As $V(G) \setminus N_G(S(M)) = \emptyset$, by Proposition 2.2, N is in one of the following cases:

(i) N has two edges one of them incident to u and the other incident to v or one of them is uv and the other is an edge not adjacent to uv.

(ii) N has only one edge which is adjacent to uv.

(iii) N is the empty matching.

So, there are $(d_G(u)-1)(d_G(v)-1)-|N_G(u)\cap N_G(v)|+s$ matchings in case (i) (note that in N the edges do not have a common vertex), there are $d_G(u) - 1 + d_G(v) - 1$ matchings in case (ii) and there are only one matching in case (iii). Consequently, the result follows. □

Example 4.2. Consider the graph of Example 3.2. As $G^{(2)}$ is the cycle C_4 we have

$$deg_G(M) = deg_{C_4}(M^{(2)}) + 5,$$

where $M^{(2)}$ has a unique edge of C_4. As all vertices of C_4 have degree 2 and there is not cycles of length 3 in C_4, by Theorem 4.1, we get $deg_{C_4}(M^{(2)}) = 2 \times 2 - 0 + 1$ (note that there is an edge in C_4 not adjacent to the edge of $M^{(2)}$). Consequently,

$$deg_G(M) = deg_{C_4}(M^{(2)}) + 5 = 5 + 5 = 10.$$

The next result is a generalization of Theorem 4.2 in [6] when M is a matching in a tree T having a single edge that is a pendant edge (see also Corollary 3.2 in [6]).

Corollary 4.3. *Let G be a connected graph and u and v be two vertices of G such that v is a pendant vertex. Let $M = \{uv\}$ be a matching in G. Then*

$$deg_G(M) = d_G(u) + s,$$

where s is the number of edges in G not adjacent to uv.

Proof. Since $d_G(v) = 1$ and $|N_G(u) \cap N_G(v)| = 0$, the result follows from Theorem 4.1. □

5 Matchings having two edges

The focus of this section is the degree of a matching M in G, with two edges. If using the elimination operation in G (see Section 3) we obtain two subgraphs each one of them with a matching having a single edge, we use the formula described in Section 4 and we get the degree of M. Therefore, in this section we assume that $V(G) \setminus N_G(S(M)) = \emptyset$. The Proposition 5.1 shows us that calculating the degree of a matching having more than two edges is complicated. Here we use other operation on G with a matching, called withdraw/choose. This operation consists on deleting two vertices and choosing a certain connected component in the obtained subgraph.

Theorem 5.1. *Let G be a connected graph and v_1, v_2, v_3, v_4 be four vertices of G. Let M be a matching in G such that $V(G) \setminus N_G(S(M)) = \emptyset$ and $M = \{v_1v_2, v_3v_4\}$. Let G^1 (respectively, G^2) be the connected component of the subgraph of G induced by the vertex set $V(G)\setminus\{v_3, v_4\}$ where v_1v_2 belongs (respectively, $V(G) \setminus \{v_1, v_2\}$ where v_3v_4 belongs) and $M^i = M \cap E(G^i)$, for $i = 1, 2$. Then*

$$deg_G(M) = deg_{G^1}(M^1) + deg_{G^2}(M^2) - s + k + \sum_{\substack{i,j=1,\\ i\neq j}}^{2} \sum_{\substack{h,p=3,\\ h\neq p}}^{4} a_{ih}[d_{G^1}(v_j)d_{G^2}(v_p) - |N_{G^1}(v_j) \cap N_{G^2}(v_p)|],$$

where s is the number of edges in G not adjacent to v_1v_2 nor to v_3v_4, k is the number of cycles of length 4, in G, where v_1v_2, v_3v_4 belong and a_{fg} is the number of edges incident to v_f and to v_g, for $1 \leq f < g \leq 4$.

Proof. Let N be a matching in G adjacent to M in the matching polytope $\mathcal{M}(G)$. By Theorem 1.1, $M\Delta N$ is an M-good path or an M-alternating cycle in G.

Since M has two edges, if $M\Delta N$ is a cycle (M-alternating cycle), then $M\Delta N$ has length 4 and contains the edges of M. Thus, there are k cycles of these kind and consequently, k matchings adjacent to M.

Suppose that $M\Delta N$ is an M-good path. Using Proposition 2.2, $|N| = 1$ or $|N| = 2$ or $|N| = 3$. Since an M-good path is an oo-M-path or a cc-M-path or an oc-M-path, we get:

1. If $|N| = 1$, then
   (a) $N = \{v_1v_2\}$ or
   (b) $N = \{v_3v_4\}$ or
   (c) the unique edge of N is adjacent to v_1v_2 and to v_3v_4.
2. If $|N| = 2$, then
   (a) $v_1v_2 \in N$ and the other edge of N is adjacent to v_3v_4 but not adjacent to v_1v_2 or
   (b) $v_3v_4 \in N$ and the other edge of N is adjacent to v_1v_2 but not adjacent to v_3v_4 or

(c) N has an edge e adjacent to v_1v_2 and to v_3v_4 and the other edge of N is not adjacent to e but it is adjacent to one of the edges of M.

3. If $|N| = 3$, then

(a) $v_1v_2 \in N$ and the other two edges of N are not adjacent to each other nor v_1v_2 but they are adjacent to v_3v_4 or

(b) $v_3v_4 \in N$ and the other two edges of N are not adjacent to each other nor v_3v_4 but they are adjacent to v_1v_2 or

(c) $\{v_1v_2, v_3v_4\} \subset N$ and the other edge of N is not adjacent to v_1v_2 nor v_3v_4 or

(d) N has an edge e adjacent to v_1v_2 and to v_3v_4 and the other two edges of N are not adjacent to each other nor to e but they are adjacent to the edges of M.

So, there are

$$2 + \sum_{i=1}^{2}\sum_{h=3}^{4} a_{ih}$$

matchings, in case 1., there are

$$d_{G^1}(v_1) - 1 + d_{G^1}(v_2) - 1 + d_{G^2}(v_3) - 1 + d_{G^2}(v_4) - 1 +$$
$$\sum_{\substack{i,j=1,\\ i\neq j}}^{2}\ \sum_{\substack{h,p=3,\\ h\neq p}}^{4} a_{ih}[(d_{G^1}(v_j) - 1) + (d_{G^2}(v_p) - 1)]$$

matchings, in case 2., there are

$$(d_{G^2}(v_3) - 1)(d_{G^2}(v_4) - 1) + (d_{G^1}(v_1) - 1)(d_{G^1}(v_2) - 1) -$$
$$|N_{G^1}(v_1) \cap N_{G^1}(v_2)| - |N_{G^2}(v_3) \cap N_{G^2}(v_4)| + s +$$
$$\sum_{\substack{i,j=1,\\ i\neq j}}^{2}\ \sum_{\substack{h,p=3,\\ h\neq p}}^{4} a_{ih}[(d_{G^1}(v_j) - 1)(d_{G^2}(v_p) - 1) - |N_{G^1}(v_j) \cap N_{G^2}(v_p)|]$$

matchings, in case 3.

Consequently,

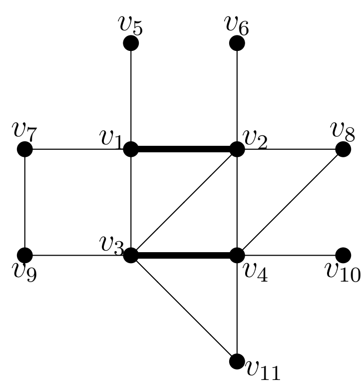
$$deg_G(M) = k + 2 + \sum_{i=1}^{2}\sum_{h=3}^{4} a_{ih} +$$
$$d_{G^1}(v_1) - 1 + d_{G^1}(v_2) - 1 + d_{G^2}(v_3) - 1 + d_{G^2}(v_4) - 1 +$$
$$\sum_{\substack{i,j=1,\\ i\neq j}}^{2}\ \sum_{\substack{h,p=3,\\ h\neq p}}^{4} a_{ih}[(d_{G^1}(v_j) - 1) + (d_{G^2}(v_p) - 1)] +$$

$$(d_{G^2}(v_3)-1)(d_{G^2}(v_4)-1)+(d_{G^1}(v_1)-1)(d_{G^1}(v_2)-1)-$$
$$|N_{G^1}(v_1)\cap N_{G^1}(v_2)|-|N_{G^2}(v_3)\cap N_{G^2}(v_4)|+s+$$
$$\sum_{\substack{i,j=1,\\ i\neq j}}^{2}\ \sum_{\substack{h,p=3,\\ h\neq p}}^{4} a_{ih}[(d_{G^1}(v_j)-1)(d_{G^2}(v_p)-1)-|N_{G^1}(v_j)\cap N_{G^2}(v_p)|]=$$

$$d_{G^1}(v_1)d_{G^1}(v_2)-|N_{G^1}(v_1)\cap N_{G^1}(v_2)|+s+$$
$$d_{G^2}(v_3)d_{G^2}(v_4)-|N_{G^2}(v_3)\cap N_{G^2}(v_4)|+s-s+$$
$$k+\sum_{\substack{i,j=1,\\ i\neq j}}^{2}\ \sum_{\substack{h,p=3,\\ h\neq p}}^{4} a_{ih}[d_{G^1}(v_j)d_{G^2}(v_p)-|N_{G^1}(v_j)\cap N_{G^2}(v_p)|].$$

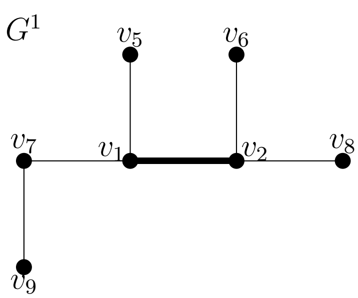
Using Theorem 4.1 we get the result. □

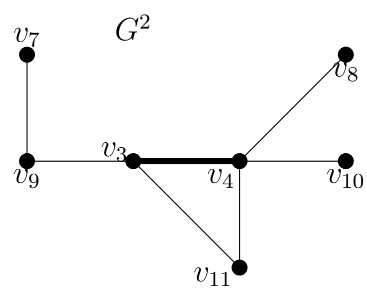
Example 5.2. Let $M=\{v_2v_3,v_3v_4\}$ (edges of M in dark) the following matching in G:



Note that $N_G(S(M))=V(G)$.

The connected component of the subgraph of G induced by the vertex set $V(G)\setminus\{v_3,v_4\}$ where v_1v_2 belongs (respectively, $V(G)\setminus\{v_1,v_2\}$ where v_3v_4 belongs) is G^1 (respectively, G^2) and $M^1=M\cap E(G^1)=\{v_1v_2\}$, $M^2=M\cap E(G^2)=\{v_3v_4\}$.





Using Theorem 5.1,

$$deg_G(M) = deg_{G^1}(M^1) + deg_{G^2}(M^2) - s + k +$$

$$\sum_{\substack{i,j=1,\\ i\neq j}}^{2} \sum_{\substack{h,p=3,\\ h\neq p}}^{4} a_{ih}[d_{G^1}(v_j)d_{G^2}(v_p) - |N_{G^1}(v_j) \cap N_{G^2}(v_p)|],$$

where s is the number of edges in G not adjacent to v_1v_2 nor to v_3v_4, k is the number of cycles of length 4, in G, where v_1v_2, v_3v_4 belong and a_{fg} is the number of edges incident to v_f and to v_g, for $1 \leq f < g \leq 4$.

Consequently, $s = 1$ (the edge v_7v_9) and $k = 1$ (the cycle $v_1v_2, v_2v_3, v_3v_4, v_4v_1$). Moreover,

$$a_{13} = 1,\ a_{14} = 0,\ a_{23} = 1,\ a_{24} = 1,$$

$$d_{G^2}(v_3) = 3,\ d_{G^2}(v_4) = 4,\ d_{G^1}(v_1) = 3,\ d_{G^1}(v_2) = 3,$$
$$|N_{G^1}(v_1) \cap N_{G^2}(v_3)| = 0,\ |N_{G^1}(v_1) \cap N_{G^2}(v_4)| = 0,$$
$$|N_{G^1}(v_2) \cap N_{G^2}(v_3)| = 0,\ |N_{G^1}(v_2) \cap N_{G^2}(v_4)| = 1$$

(note that $\{v_8\} = N_{G^1}(v_2) \cap N_{G^2}(v_4)$). So, by Theorems 4.1 and 5.1,

$$deg_G(M) =$$

$$(9 - 0 + 1) + (12 - 1 + 1) - 1 + 1 + a_{13}(12 - 1) + a_{23}(12 - 0) + a_{24}(9 - 0) =$$
$$10 + 12 + 11 + 12 + 9 = 54.$$

We finish this section with the degree of a matching with two edges in C_4.

Corollary 5.3. *Let M be a matching in C_4 with two edges. Then*

$$deg_{C_4}(M) = 5.$$

Proof. Let v_1, v_2, v_3, v_4 be the four vertices of C_4 and $M = \{v_1v_2, v_3v_4\}$. Note that C_4 has a unique cycle, the connected component of the subgraph of C_4 induced by the two vertices incident in an edge of M is P_2 and the restriction of M to this subgraph has only the edge of P_2. Moreover, all vertices of P_2 has degree 1 and there are not edges in C_4 not adjacent to the both edges of M. Consequently, using Theorem 5.1 we get

$$deg_{C_4}(M) = deg_{P_2}(M^1) + deg_{P_2}(M^2) + 1 + \sum_{i=1}^{2}\sum_{h=3}^{4} a_{ih} = 1 + 1 + 1 + 2 = 5.$$

□

6 Computing the degree of a matching

The goal of this section is to compute the degree of a matching with two or more edges. As in [6] this question was solved for trees, we will consider graphs with cycles. In this section we only use the withdraw operation on the graph with a matching.

Let $G = (V(G), E(G))$ be a connected graph, $I \subset V(G)$ and $J \subset E(G)$. We denote by $G^{(I)}$ the subgraph of G induced by the vertex set $V(G) \setminus I$. Let M be a matching in G. The matching M gives rise to a matching $M^{(I)} = M \cap E(G^{(I)})$ in $G^{(I)}$ and to a matching $M \setminus J$ in G.

Theorem 6.1. *Let G be a connected graph having at least one cycle. Let uv and zw be two different edges of the matching M in G such that there is no path P in G contained uv and zw with $V(P) \cap S(M) = V(P)$. If $V(G) \setminus N_G(S(M)) = \emptyset$, then*

$$deg_G(M) =$$

$$deg_{G^{(\{u,v\})}}(M^{(\{u,v\})}) + deg_{G^{(\{z,w\})}}(M^{(\{z,w\})}) - deg_{G^{(\{u,v,z,w\})}}(M^{(\{u,v,z,w\})}).$$

Proof. Let N be a matching in G adjacent to M in the matching polytope $\mathcal{M}(G)$. Then the matching N verifies one of the following cases:

(i) the edge uv is in N

(ii) the edge zw is in N

(iii) neither the edge uv nor the edge zw are in N.

Then, we get

(1) N is a matching, in G, adjacent to M and verifies case (i) if and only if $N \setminus \{uv\}$ is a matching in $G^{(\{u,v\})}$ adjacent to $M \setminus \{uv\} = M^{(\{u,v\})}$,

(2) N is a matching, in G, adjacent to M and verifies case (ii) if and only if $N \setminus \{zw\}$ is a matching in $G^{(\{z,w\})}$ adjacent to $M \setminus \{zw\} = M^{(\{z,w\})}$.

If the matching N have both edges, uv and zw, then N is a matching that verifies cases (i) and (ii). So, if N contains both edges, uv and zw, and it is adjacent to M, then $N \setminus \{uv, zw\}$ is a matching in $G^{(\{u,v,z,w\})}$ adjacent to $M \setminus \{uv, zw\} = M^{(\{u,v,z,w\})}$. Moreover, the converse is true.

By Theorem 1.1, $M \Delta N$ is an M-good path or an M-alternating cycle in G. If N is a matching and verifies case (iii), then uv and zw are edges of $M \Delta N$. This implies that there is a path P in G contained uv and zw with $V(P) \cap S(M) = V(P)$, contradicting the hypothesis. Thus, there is no matching N in G adjacent to M and verifying the case (iii).

Therefore, the result follows. □

Remark 6.2. The last result transforms the degree of a matching with some constraints in a sum of degree of matchings having fewer edges than the initial one.

7 Conclusions

In this paper we consider graphs having cycles and we presented formulas to compute the degree of a matching using some operations on the initial graph. More precisely, firstly we focus on matchings with vertices that are not in the neighbor of the saturated vertices by the matching. Then our goal was the matchings with one and two edges. And in Section 6 we show a formula to obtain the degree of a matching with some constraints. We also defined two operations on the graph with a matching, the elimination and the withdraw/choose.

An open question is how to compute the degree of a matching without the constraints present in Theorem 6.1.

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