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Rank of signed cacti

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Abstract

A signed cactus $\dot{G}$ is a connected signed graph such that every edge belongs to at most one cycle. The rank of $\dot{G}$ is the rank of its adjacency matrix. In this paper we prove that

$$\sum_{i=1}^{k} n_i - 2k \leq \operatorname{rank}(\dot{G}) \leq \sum_{i=1}^{k} n_i - 2t + 2s,$$

where k is the number of cycles in $\dot{G}$, $n_1, n_2, \dots, n_k$ are their lengths, t is the number of cycles whose rank is their order minus two, and s is the number of edges outside cycles. Signed cacti attaining the lower bound are determined.

1 Introduction

A *signed graph* $\dot{G}$ is a finite, undirected graph without loops or parallel edges, in which every edge has been declared positive or negative. The adjacency matrix $A_{\dot{G}}$ is an $n \times n$ matrix whose (i, j)-entry is 1 if the vertices i and j are joined by a positive edge, -1 if they are joined by a negative edge, and 0 if they are non-adjacent. The *order* n of $\dot{G}$ is the number of vertices of $\dot{G}$. The *rank* $\operatorname{rank}(\dot{G})$ of $\dot{G}$ is the rank of $A_{\dot{G}}$.

A *signed cactus* (also known as a *tree-like signed graph*) $\dot{G}$ is a connected signed graph such that every pair of induced cycles (if any) has at most one common vertex; equivalently, every edge of $\dot{G}$ belongs to at most one cycle.

Inspired by the result of Chen and Tian [2] that offers a lower bound for the rank of the skew adjacency matrix of an oriented cactus, in this contribution we prove a similar lower bound for the rank of a signed cactus, along with an upper bound. Signed cacti attaining the lower bound are determined, as well. Our results can be seen in the light of recently established links between the spectrum of an oriented graph and the skew spectrum of a related signed graph (with a precise meaning of 'related') [5]. According to these links, there

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is a strong relationship between the theory of skew spectra of oriented graphs and the theory of spectra of signed graphs, and therefore the similarity between the lower bound of [2] and the lower bound obtained in this paper is unsurprising.

In the next section we go straight to the results.

2 Results

All the necessary terminology and notation are given in this paragraph. We write I, O and $\mathbf{0}$ to denote the identity matrix, the all-zero matrix and the all-zero vector, respectively. For a vertex v of a signed graph $\dot{G}$, $\dot{G}-v$ denotes the induced subgraph obtained by removing v. Similarly, if $\dot{H}$ is an induced subgraph of $\dot{G}$, then $\dot{G}-\dot{H}$ denotes the induced subgraph obtained by removing the vertices of $\dot{H}$. In this spirit, $\dot{H}+v$ is an induced subgraph whose vertex set is the union of the vertex set of $\dot{H}$ and a single vertex v. A *cut-vertex* of a connected signed graph is a vertex whose removal results in a disconnected signed graph. A *pendant vertex* is a vertex of degree one. A cycle in $\dot{G}$ is *even* (resp. *odd*) if its order is even (odd). A cycle is *positive* if the product of its edge signs is 1. Otherwise, it is *negative*.

Some known results are needed.

Lemma 2.1 (cf. [1, Theorem 3]). *For a vertex v of a signed graph $\dot{G}$,* $\operatorname{rank}(\dot{G}-v) \leq \operatorname{rank}(\dot{G}) \leq \operatorname{rank}(\dot{G}-v)+2$.

The next results reveals a situation in which the previous upper bound is attained.

Lemma 2.2 ([3]). *Let v be a cut-vertex of a connected signed graph $\dot{G}$ and $\dot{H}$ a disjoint union of some connected components of $\dot{G}-v$. If* $\operatorname{rank}(\dot{H}+v)=\operatorname{rank}(\dot{H})+2$, *then* $\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{G}-v)+2$.

We include a sketched proof, while for details we refer the reader to [3]. If $\dot{H}$ includes all the components of $\dot{G}-v$, there is nothing to prove. Otherwise, we write

$$A_{\dot{G}}=\begin{pmatrix} A_{\dot{H}} & \mathbf{x} & O \\ \mathbf{x}^\intercal & 0 & \mathbf{y}^\intercal \\ O & \mathbf{y} & A_{\dot{G}-v-\dot{H}} \end{pmatrix}, \tag{2.1}$$

and observe that, under the assumption of the lemma, $\mathbf{x}$ is not in the range (i.e., the column space) of $A_{\dot{H}}$ and a basis of the 2×2 top-left block of $A_{\dot{G}}$ is spanned by $\begin{pmatrix}\mathbf{c}_i\\0\end{pmatrix}$, $\begin{pmatrix}\mathbf{x}\\0\end{pmatrix}$ and $\begin{pmatrix}\mathbf{0}\\1\end{pmatrix}$, where $\{\mathbf{c}_i\}$ is the basis of the column space of $A_{\dot{H}}$. Then

$$\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{H})+1+\operatorname{rank}\begin{pmatrix}\mathbf{0} & O\\ 1 & \mathbf{y}^\intercal\\ \mathbf{0} & A_{\dot{G}-v-\dot{H}}\end{pmatrix}=\operatorname{rank}(\dot{H})+2+\operatorname{rank}(\dot{G}-v-\dot{H}),$$

which gives the desired assertion, as $\dot{G}-v$ is a disjoint union of $\dot{H}$ and $\dot{G}-v-\dot{H}$, with $\operatorname{rank}(\dot{G}-v)=\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-v-\dot{H})$.

We proceed with the following lemma.

Lemma 2.3. *Let v be a cut-vertex of a connected signed graph $\dot{G}$ and $\dot{H}$ a disjoint union of some connected components of $\dot{G}-v$. If $\operatorname{rank}(\dot{H}+v)=\operatorname{rank}(\dot{H})$, then $\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})$.*

Proof. The result follows directly if $\dot{H}$ includes all the components of $\dot{G}-v$, i.e., $\dot{H}\cong\dot{G}-v$. Otherwise, if $\operatorname{rank}(\dot{H}+v)=\operatorname{rank}(\dot{H})$, then the vector $\mathbf{x}$ of (2.1) belongs to the range of $A_{\dot{H}}$, giving $A_{\dot{H}}\mathbf{z}=\mathbf{x}$, for $\mathbf{z}\neq\mathbf{0}$. In addition, $\mathbf{x}^\intercal\mathbf{z}=0$, as otherwise yields $\operatorname{rank}(\dot{H})<\operatorname{rank}(\dot{H}+v)$. With consistent submatrix sizes, this gives the following rank preserving transformation of $A_{\dot{G}}$:

$$\begin{pmatrix} I & -\mathbf{z} & O\\ \mathbf{0}^\intercal & 1 & \mathbf{0}^\intercal\\ O & \mathbf{0} & I\end{pmatrix}^\intercal \cdot A_{\dot{G}}\cdot \begin{pmatrix} I & -\mathbf{z} & O\\ \mathbf{0}^\intercal & 1 & \mathbf{0}^\intercal\\ O & \mathbf{0} & I\end{pmatrix}=$$
$$\begin{pmatrix} I & \mathbf{0} & O\\ -\mathbf{z}^\intercal & 1 & \mathbf{0}^\intercal\\ O & \mathbf{0} & I\end{pmatrix}\begin{pmatrix} A_{\dot{H}} & \mathbf{x} & O\\ \mathbf{x}^\intercal & 0 & \mathbf{y}^\intercal\\ O & \mathbf{y} & A_{\dot{G}-v-\dot{H}}\end{pmatrix}\begin{pmatrix} I & -\mathbf{z} & O\\ \mathbf{0}^\intercal & 1 & \mathbf{0}^\intercal\\ O & \mathbf{0} & I\end{pmatrix}=\begin{pmatrix} A_{\dot{H}} & \mathbf{0} & O\\ \mathbf{0}^\intercal & 0 & \mathbf{y}^\intercal\\ O & \mathbf{y} & A_{\dot{G}-v-\dot{H}}\end{pmatrix},$$

which leads to the desired assertion. □

A similar proof technique can be found in [2]. We continue with the following theorem, a crucial ingredient for the proof of the forthcoming Theorem 2.6.

Theorem 2.4. *Let v be a cut-vertex of a connected signed graph $\dot{G}$ and $\dot{H}$ a disjoint union of some connected components of $\dot{G}-v$. The following statements hold true.*

(i) *If* $\operatorname{rank}(\dot{H}+v)=\operatorname{rank}(\dot{H})+2$, *then* $\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})\leq\operatorname{rank}(\dot{G})\leq\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})+2$.

(ii) *If* $\operatorname{rank}(\dot{H}+v)=\operatorname{rank}(\dot{H})+1$, *then* $\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})-1\leq\operatorname{rank}(\dot{G})\leq\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})+1$.

Proof. As in the previous proof, the case $\dot{H}\cong\dot{G}-v$ is resolved directly. We assume that $\dot{H}\ncong\dot{G}-v$, and write $\dot{F}$ for the signed graph obtained by removing the vertex v from $\dot{G}-\dot{H}$. Since $\dot{G}-v$ is a disjoint union of $\dot{H}$ and $\dot{F}$, we have

$$\operatorname{rank}(\dot{G}-v)=\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{F}). \tag{2.2}$$

Due to Lemma 2.1, we have

$$\operatorname{rank}(\dot{F})\leq\operatorname{rank}(\dot{G}-\dot{H})\leq\operatorname{rank}(\dot{F})+2. \tag{2.3}$$

(i): For $\operatorname{rank}(\dot{H}+v)=\operatorname{rank}(\dot{H})+2$, Lemma 2.2 in conjunction with (2.2) yields $\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{F})+2$. Employing (2.3), we obtain $\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})\leq\operatorname{rank}(\dot{G})\leq\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})+2$, as desired.

(ii): We assume that $\operatorname{rank}(\dot{H}+v)=\operatorname{rank}(\dot{H})+1$ and consider the three possibilities for $\operatorname{rank}(\dot{G}-\dot{H})$.

First, if $\operatorname{rank}(\dot{G}-\dot{H})=\operatorname{rank}(\dot{F})+2$, by substituting $\dot{F}$ for $\dot{H}$ in Lemma 2.2, we obtain $\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{G}-v)+2=\operatorname{rank}(\dot{F})+\operatorname{rank}(\dot{H})+2$, where the latter equality follows from (2.2). This gives

$$\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H}). \tag{2.4}$$

Let $\operatorname{rank}(\dot{G}-\dot{H})=\operatorname{rank}(\dot{F})+1$. From Lemma 2.1 and (2.2), we obtain $\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{F})\leq\operatorname{rank}(\dot{G})\leq\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{F})+2$, which together with the previous equality leads to

$$\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})-1\leq\operatorname{rank}(\dot{G})\leq\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})+1. \tag{2.5}$$

Finally, assume that $\operatorname{rank}(\dot{G}-\dot{H})=\operatorname{rank}(\dot{F})$. By substituting $\dot{F}$ for $\dot{H}$ in Lemma 2.3, we arrive at $\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{H}+v)+\operatorname{rank}(\dot{F})=\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{F})+1$, where the second equality follows from the assumption of item (ii) of this statement. It follows

$$\operatorname{rank}(\dot{G})=\operatorname{rank}(\dot{H})+\operatorname{rank}(\dot{G}-\dot{H})+1. \tag{2.6}$$

The desired inequalities follow by taking into account (2.4)–(2.6). □

The next result is needed, as well. It can be proved directly, but we also provide some references.

Lemma 2.5 ([3, 4])**.** *The following statements hold true.*

(i) For a signed path $\dot{P}_n$ of order n,

$$\operatorname{rank}(\dot{P}_n)=\begin{cases} n & \text{if } n \text{ is even},\\ n-1 & \text{if } n \text{ is odd}.\end{cases}$$

(ii) For a signed cycle $\dot{C}_n$ of order n,

$$\operatorname{rank}(\dot{C}_n)=\begin{cases} n-2 & \text{if either } n\cong 0 \pmod 4 \text{ and } \dot{C}_n \text{ is positive or}\\ & n\cong 2 \pmod 4 \text{ and } \dot{C}_n \text{ is negative},\\ n & \text{otherwise}.\end{cases}$$

Here is the main result of this paper.

Theorem 2.6. *Let $\dot{G}$ be a signed cactus with k cycles of lengths $n_1, n_2, \dots, n_k$, such that exactly t of them are even and positive (resp. negative) if their order is (is not) divisible by 4. Assume that exactly s edges do not belong to any cycle. Then*

$$\sum_{i=1}^{k} n_i-2k\leq\operatorname{rank}(\dot{G})\leq\sum_{i=1}^{k} n_i-2t+2s.$$

Proof. For $k = 0$, the lower bound reduces to 0, and the upper bound reduces to $2(n-1)$, where n is the order of $\dot{G}$, and so both inequalities hold. Let $k \geq 1$.

We first consider the lower bound. Observe that the existence of pendant vertices leaves the bound unchanged and may increase the rank, and thus we assume that $\dot{G}$ has no pendant vertices. For $k = 1$, the result follows from Lemma 2.5(ii). We proceed by the induction argument, i.e., we assume that the statement holds for every signed cactus with $k-1$ cycles, and consider $\dot{G}$ having k ($k \geq 2$) cycles. Let $\dot{C}_{n_k}$ be a cycle with exactly one vertex, say v, of degree ≥ 3. Clearly, such a cycle exists under the assumption that $\dot{G}$ has no pendant vertices.

If $\mathrm{rank}(\dot{C}_{n_k}) = n_k - 2$, then by Lemma 2.5(i) we have $\mathrm{rank}(\dot{C}_{n_k} - v) = n_k - 2$, as $\dot{C}_{n_k} - v$ is a path of an odd order. From Lemma 2.3 with $\dot{C}_{n_k} - v$ in the role of $\dot{H}$, we obtain $\mathrm{rank}(\dot{G}) = \mathrm{rank}(\dot{C}_{n_k} - v) + \mathrm{rank}(\dot{G} - \dot{C}_{n_k} + v)$, which under the induction hypothesis leads to $\mathrm{rank}(\dot{G}) \geq n_k - 2 + \sum_{i=1}^{k-1} n_i - 2(k-1) = \sum_{i=1}^{k} n_i - 2k$.

In the light of Lemma 2.5(i), $\mathrm{rank}(\dot{C}_{n_k}) = n_k$ implies

$$\mathrm{rank}(\dot{C}_{n_k} - v) = \begin{cases} n_k - 2 & \text{if } n_k \text{ is even,} \\ n_k - 1 & \text{if } n_k \text{ is odd.} \end{cases}$$

For n_k even, Theorem 2.4(i) leads to $\mathrm{rank}(\dot{G}) \geq \mathrm{rank}(\dot{C}_{n_k} - v) + \mathrm{rank}(\dot{G} - \dot{C}_{n_k} + v) = n_k - 2 + \mathrm{rank}(\dot{G} - \dot{C}_{n_k} + v)$. After employing the induction hypothesis, we arrive at the desired inequality.

For n_k odd, Theorem 2.4(ii) leads to $\mathrm{rank}(\dot{G}) \geq \mathrm{rank}(\dot{C}_{n_k} - v) + \mathrm{rank}(\dot{G} - \dot{C}_{n_k} + v) - 1 = n_k - 2 + \mathrm{rank}(\dot{G} - \dot{C}_{n_k} + v)$, and the result follows.

Consider now the upper bound. For $k = 1$, the result follows from Lemma 2.5(ii) and Lemma 2.1. Indeed, the rank of the unique cycle is given in the former lemma, and the latter one tells us that adding a pendant vertex increases the rank by at most 2. We proceed as in the previous case and assume that $\dot{G}$ has k ($k \geq 2$) cycles, along with the hypothesis that the statement holds for every signed cactus with $k-1$ cycles. Suppose first that $\dot{G}$ has a cycle, say $\dot{C}_{n_k}$, with exactly one vertex, say again v, of degree ≥ 3.

If $\mathrm{rank}(\dot{C}_{n_k}) = n_k - 2$ (which occurs exactly if $\dot{C}_{n_k}$ is one of the t cycles given in the statement formulation), then by virtue of Lemma 2.3, we obtain $\mathrm{rank}(\dot{G}) = \mathrm{rank}(\dot{C}_{n_k} - v) + \mathrm{rank}(\dot{G} - \dot{C}_{n_k} + v)$, and then Lemma 2.5(i) yields $\mathrm{rank}(\dot{G}) \leq n_k - 2 + \sum_{i=1}^{k-1} n_i - 2(t-1) + 2s = \sum_{i=1}^{k} n_i - 2t + 2s$.

If $\mathrm{rank}(\dot{C}_{n_k}) = n_k$, the desired inequality is obtained by a slight modification of the proof for the lower bound; the only difference is that now we use the upper bounds of Theorem 2.4.

If $\dot{C}_{n_k}$ has trees attached at its vertices, which contain r edges in total, then $\mathrm{rank}(\dot{G}) \leq \sum_{i=1}^{k} n_i - 2t + 2(s-r) + 2r$, as follows by multiple application of Lemma 2.5 to the vertices of trees. The latter inequality concludes the entire proof. □

We reformulate the previous result to the following corollary suggested by the referee.

Corollary 2.7. *Let $\dot{G}$ be a signed cactus of Theorem 2.6. Then*

$$m - s - 2k \leq \mathrm{rank}(\dot{G}) \leq m - 2t + s,$$

where m is the number of edges of $\dot{G}$.

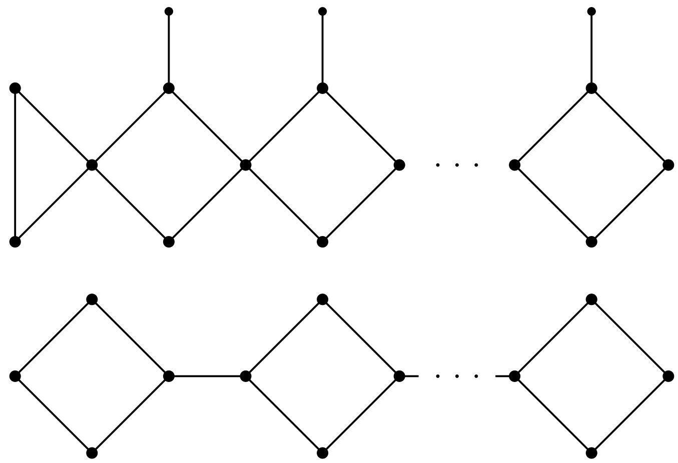

Figure 1: Examples of signed cacti attaining the upper bound of Theorem 2.6. In these examples, all edges are positive.

This result, of course, follows from $m = \sum_{i=1}^{k} n_i + s$. Consider now the equality in the lower bound of Theorem 2.6.

Theorem 2.8. *The lower bound of Theorem 2.6 is attained if and only if either $\dot{G}$ is an isolated vertex or each of its edges belongs to a cycle, and every cycle is even and positive (resp. negative) if its order is (is not) divisible by* 4.

Proof. Assume first that $\dot{G}$ is as in the statement formulation. If it consists of a single vertex, then the bound is obviously attained. In the remaining case, we again proceed with the induction. For $k = 1$, $\dot{G}$ is an even cycle and the result follows from Lemma 2.5(ii). If $\dot{C}_{n_k}$ is a cycle with exactly one vertex, say v, of degree ≥ 3, then $\mathrm{rank}(\dot{C}_{n_k}) = \mathrm{rank}(\dot{C}_{n_k} - v)$ (by Lemma 2.5), and the result follows by employing Lemma 2.3 (with $\dot{C}_{n_k} - v$ in the role of $\dot{H}$) and the induction hypothesis.

Assume now that the equality is attained. We first eliminate te possibility that $\dot{G}$ contains an odd cycle. If $\dot{C}_{n_1}$ is such a cycle, then we have $\mathrm{rank}(\dot{C}_{n_1}) = n_1$. We proceed to build $\dot{G}$ by a successive addition of pendant vertices or cycles. Adding a vertex does not decrease the rank, and adding the entire cycle of length q increases it by at least $q-2$, as follows by Theorem 2.4. Therefore, the rank of $\dot{G}$ is at least $n_1+\sum_{i=2}^{k} n_i-2(k-1) = \sum_{i=1}^{k} n_i-2(k-1) > \sum_{i=1}^{k} n_i-2k$.

The remainder of the proof follows from Theorem 2.6(i) of [5] which states that the rank of a bipartite signed graph equals the rank of an associated oriented graph (with a precise meaning of 'associated') and Theorem 1.2 of [2] stating that the corresponding lower bound for oriented graphs is attained if and only if they are associated with signed graphs given in the formulation of this statement. □

Observe that the upper bound of Theorem 2.6 may be trivial in the sense that is larger than the order of $\dot{G}$. For $t = 0$, since every vertex belonging to a cycle in $\dot{G}$ is counted at least once in the sum of the upper bound, and every vertex not belonging to any cycle is incident with at least one edge not belonging to cycles, we obtain that the upper bound is not less than the order of $\dot{G}$, and $\dot{G}$ attains the bound if and only if it is an isolated vertex or a signed cycle; since $t = 0$, this cycle has restrictions according to the theorem. For $t \geq 1$, we got a rather complicated situation, and the question of equality remains a challenging problem. Our experiments show that, for example, two infinite families of signed graphs illustrated in Fig. 1 attain the bound. The same holds if we remove pendant vertices form the first one. Further examples are constructed by changing lengths of cycles with a caveat that their rank must remain unchanged. This in particular means that the triangle of the first example can be replaced by a negative cycle of length $4k + 2$ ($k \geq 1$).

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