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Research Article

Eigenvalues of multipart matrices and their applications

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Abstract

A square matrix is called a multipart matrix if all its diagonal entries are zero and all other entries in each column are constant. In this paper, we describe various interesting spectral properties of multipart matrices. We provide suitable bounds for the spectral radius of a multipart matrix. Later on, we show applications of multipart matrices in spectral graph theory.

1 Introduction

A graph $G = (V, E)$ is called a complete k-partite graph if its vertex set is partitioned as $V = V_1 \cup V_2 \cup \cdots \cup V_k$ such that two vertices u and v are adjacent if and only if $u \in V_i$ and $u \in V_j$ where $i \neq j$, $1 \leq i, j \leq k$. A complete k-partite graph with partition sizes $p_1, p_2, \ldots, p_k$ is denoted by $K_{p_1,p_2,\cdots,p_k}$. For a complete k-partite graph G with partition sizes $p_1, p_2, \ldots, p_k$, the vertex set partition $V_1 \cup V_2 \cup \cdots \cup V_k$ consisting of independent subsets of G is an equitable partition. The quotient matrix Q for that equitable partition is given by

$$Q = \begin{bmatrix} 0 & p_2 & p_3 & \cdots & p_k \\ p_1 & 0 & p_3 & \cdots & p_k \\ p_1 & p_2 & 0 & \cdots & p_k \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ p_1 & p_2 & p_3 & \cdots & 0 \end{bmatrix}.$$

Let A denote the $(0, 1)$-adjacency matrix of $K_{p_1,p_2,\cdots,p_k}$. Then every eigenvalue of Q is also an eigenvalue of A [5, Theorem 9.3.3]. Let G be a simple, connected, finite graph and A be the $(0, 1)$-adjacency matrix of G. Let D be the diagonal matrix whose i-th diagonal entry is the degree of the vertex i. The matrix $\mathcal{A} = D^{-1}A$ is called the normalized adjacency matrix of G. The matrix $\mathcal{A}$ is similar to the matrix $D^{-\frac{1}{2}}AD^{-\frac{1}{2}}$, known as the Randić matrix

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[2]. We observe that the quotient matrix of $\mathcal{A}$ for the complete multipartite graph $K_{p_1,p_2,\cdots,p_k}$ is similar to

$$\mathcal{Q} = \begin{bmatrix} 0 & \frac{p_2}{n-p_2} & \frac{p_3}{n-p_3} & \cdots & \frac{p_k}{n-p_k} \\ \frac{p_1}{n-p_1} & 0 & \frac{p_3}{n-p_3} & \cdots & \frac{p_k}{n-p_k} \\ \frac{p_1}{n-p_1} & \frac{p_2}{n-p_2} & 0 & \cdots & \frac{p_k}{n-p_k} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{p_1}{n-p_1} & \frac{p_2}{n-p_2} & \frac{p_3}{n-p_3} & \cdots & 0 \end{bmatrix}.$$

Inspired by this special form of quotient matrices, we define a special class of matrices and call them multipart matrices.

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite sequence of real numbers. The multipart matrix $A(P) = [a_{ij}]$ corresponding to the sequence P is an $n \times n$ matrix whose entries are given by

$$a_{ij} = \begin{cases} p_j & \text{if } i \neq j, \\ 0 & \text{if } i = j. \end{cases}$$

In other words, $A(P)$ is a $n \times n$ matrix whose j-th column is $p_j(\boldsymbol{e} - \boldsymbol{e}_j)$, where $\boldsymbol{e}$ is the column vector of all ones and $\boldsymbol{e}_j$ is the j-th standard basis element of $\mathbb{R}^n$. It is easy to verify that the eigenvalues of $A(P)$ do not depend on the arrangement of the entries in the sequence P. Thus, without loss of generality, we consider the sequence P to be non-decreasing and nonzero. Since the quotient matrices for both adjacency and normalized adjacency matrices of complete multipartite graphs are special cases of multipart matrices, the study of eigenvalues for multipart matrices seems significant and interesting.

There are several papers focused on adjacency eigenvalues of complete multipartite graphs. Esser and Harary [4] discussed very interesting properties of the adjacency matrix of complete multipartite graphs. In [1, 8] authors characterized complete multipartite graphs having integral adjacency eigenvalues. Bapat and Karimi [1] provided examples of non-isomorphic complete multipartite graphs with the same spectrum. Delorme obtained [3] an explicit formula for the characteristic polynomial for complete multipartite graphs. Also, he used optimization techniques to find the eigenvalues of a class of complete multipartite graphs. Stevanović *et al.* [7] derived results related to the spectral radius and energy of complete multipartite graphs. In this paper, we show that all these results can be easily derived by considering the quotient matrix of a complete multipartite graph as a special case of multipart matrices and taking P as a finite sequence of natural numbers. On the other hand, the normalized adjacency eigenvalues of complete multipartite graphs were not much studied.

In this paper, we first investigate the spectral properties of multipart matrices. Using them, we show that some of the existing results of [3, 4] for the adjacency matrix of complete multipartite graphs can be explained very easily. Later on, we establish spectral properties for the normalized adjacency matrix of complete multipartite graphs. We also provide lower and upper bounds for the largest adjacency eigenvalue of multipart matrices, that provides suitable bounds for the spectral radius of complete multipartite graphs.

2 Eigenvalues of multipart matrices

In this section, we discuss various spectral properties of multipart matrices. For a finite sequence $P = \{p_1, p_2, \ldots, p_n\}$, we show that some of the eigenvalues $A(P)$ can be determined directly from the structure of P. We call such an eigenvalue a regular eigenvalue of $A(P)$, otherwise we call it a non-regular eigenvalue of $A(P)$. We start with a determinant formula.

Theorem 2.1. *Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite sequence of real numbers. Then*

$$\det A(P) = (-1)^{n-1}(n-1)p_1p_2 \cdots p_n.$$

Proof. We observe that,

$$\det A(P) = p_1p_2 \cdots p_n \det(J_n - I_n),$$

where J_n is the all-ones square matrix of order n. Since $J_n - I_n$ has eigenvalues -1 (with multiplicity $n-1$ and $n-1$ (with multiplicity 1), we get the required result. □

Corollary 2.2. *If $p_i \neq 0$ for all $i = 1, 2, \ldots, n$, then $A(P)$ is nonsingular and consequently, 0 is not an eigenvalue of $A(P)$.*

Theorem 2.3. *Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite sequence of real numbers, and let $\Phi_P(x) = x^n + c_1x^{n-1} + \cdots + c_n$ be the characteristic polynomial of $A(P)$. Then*

$$c_i = -(i-1) \sum_S \prod_{j \in S} p_j,$$

where the summation runs over all subsets S of $\{1, 2 \ldots, n\}$ of order i.

Proof. Since trace $A(P)$ is 0, the above equation holds for $i = 1$. For $i \geq 2$, we consider the following formula:

$$c_i = (-1)^i(\textit{sum of } i \times i \textit{ principal minors}).$$

Since an $i \times i$ principal minor of $A(P)$ is also a multipart matrix of order i, hence the result follows by Theorem 2.1. □

Theorem 2.4. *If $p_i > 0$ for all $1 \leq i \leq n$, then the eigenvalues of $A(P)$ are all real.*

Proof. Let $D = diag[p_1, p_2, \ldots, p_n]$. Then the matrix $D^{\frac{1}{2}}A(P)D^{-\frac{1}{2}}$ is a symmetric matrix and similar to $A(P)$. Hence the eigenvalues of $A(P)$ are real. □

Remark 2.5. If $p_i < 0$ for some i, then the above result may not hold. Consider $P = \{1, -1\}$. Then

$$A(P) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

whose eigenvalues are ±i.

Theorem 2.6. *Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite sequence of non-decreasing real numbers. If $p_1 = p_2 = \cdots = p_l = p \neq p_{l+1}$, then $-p$ is an eigenvalue of $A(P)$ with multiplicity exactly $l-1$.*

Proof. For all $i \geq 1$, we define column vectors in $\mathbb{R}^n$ as follows:

$$X_i = \sum_{k=1}^{i} \boldsymbol{e}_k - i\boldsymbol{e}_{i+1}.$$

Then for $i \neq j$, $X_i^T X_j = 0$. So the set $\{X_1, X_2, \ldots, X_{l-1}\}$ is orthogonal. Also note that,

$$A(P)X_i = -pX_i \quad \textit{for all } 1 \leq i \leq l-1.$$

Which implies that the multiplicity of the eigenvalue $-p$ is at least $l-1$. Let the multiplicity of $-p$ be greater than $l-1$. Then there exists a nonzero vector Y whose j-th component is nonzero for some $j > l$ and it satisfies the eigenvalue equation $A(P)Y = -pY$. Then there exist real numbers c and d such that the vector $cX_1 + dY$ has the first and j-th entries equal to 1. Now $cX_1 + dY$ is also an eigenvector corresponding to 1. Therefore, comparing the first and j-th equations of $A(cX_1 + dY) = -p(cX_1 + dY)$ we conclude that $p = p_j$. Which is a contradiction. Hence the multiplicity of $-p$ is $l-1$. □

Theorem 2.7. *If all p_i's are distinct and positive, then $A(P)$ has n distinct eigenvalues. Moreover, if $\lambda_n < \lambda_{n-1} < \cdots < \lambda_1$ are eigenvalues of $A(P)$, then $\lambda_1 > 0 > \lambda_2$.*

Proof. Let λ be an eigenvalue of $A(P)$ with multiplicity > 1. By Theorem 2.4, $A(P)$ is diagonalizable. Then there exist linearly independent eigenvectors $X, Y \in \mathbb{R}^n$ corresponding to the eigenvalue λ. Since X and Y are linearly independent, there exists $c, d \in \mathbb{R}$ such that $cX + dY$ has two components equal to 1. Suppose the i-th and j-th components of $cX + dY$ are equal to 1. Since X and Y are eigenvectors corresponding to the same eigenvalue λ, we have

$$A(cX + dY) = \lambda(cX + dY).$$

Now, comparing the i-th and j-th components, we get $p_i = p_j$. Which is a contradiction. Therefore, all eigenvalues are distinct.

Now, since $p_i > 0$ for all $i = 1, 2, \ldots, n$, $A(P)$ is irreducible and non-negative. Hence $\lambda_1 > 0$ by Perron-Frobenius theorem ([6], Theorem 8.4.4). Again, $A(P)$ is similar to $D^{\frac{1}{2}}A(P)D^{-\frac{1}{2}} = D^{\frac{1}{2}}(J_n - I_n)D^{\frac{1}{2}}$. Therefore, $\lambda_2 < 0$ by Sylvester law of inertia ([6], Theorem 4.5.8). Therefore $\lambda_2 < 0 < \lambda_1$. Which completes the proof. □

Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite sequence of real numbers. Suppose $p_1 = p_2 = \cdots = p_l = p$. Let P' be the sequence $\{lp, p_{l+1}, \ldots, p_n\}$. We construct an $(n-l+1) \times (n-l+1)$ matrix defined by

$$C(P') = A(P') + \operatorname{diag}[(l-1)p, 0, 0, \ldots, 0].$$

Then, we have the following result:

Theorem 2.8. *Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite sequence of non-decreasing real numbers and $p_1 = p_2 = \cdots = p_l = p \neq p_{l+1}$. If $P' = \{lp, p_{l+1}, \ldots, p_n\}$, then*

(a) *any eigenvalue of $C(P')$ is also an eigenvalue of $A(P)$, and*

(b) *$-p$ is not an eigenvalue of $C(P')$.*

Proof. (a) Let λ be an eigenvalue of $C(P')$ and $X = [x_1 \ x_2 \ \cdots \ x_{n-l+1}]^T$ be a corresponding eigenvector. Define $Y = [y_1 \ y_2 \ \cdots \ y_n]^T$ by

$$y_i = \begin{cases} x_1 & \text{if } i \leq l, \\ x_{i-l+1} & \text{if } l < i \leq n. \end{cases}$$

Then Y is a nonzero vector in $\mathbb{R}^n$ and $A(P)Y = \lambda Y$. Therefore λ is also an eigenvalue of $A(P)$.

(b) Let $-p$ be an eigenvalue of $C(P')$ and $x = [x_1 \ x_2 \ \cdots \ x_{n-l+1}]^T$ an eigenvector corresponding to $-p$. Then the nonzero vector $Y = [y_1 \ y_2 \ \cdots \ y_n]^T$ defined by

$$y_i = \begin{cases} x_1 & \text{if } i \leq l, \\ x_{i-l+1} & \text{if } l < i \leq n, \end{cases}$$

satisfies $A(P)Y = \lambda Y$. Now since the first l components of Y are constants, $X_i^T Y = 0$ for all $1 \leq i \leq l-1$. Which implies that, if $-p$ is an eigenvalue of $C(P')$, then the multiplicity of $-p$ as an eigenvalue of $A(P)$ is greater than $l-1$. Which contradicts Theorem 2.6. Therefore, $-p$ is not an eigenvalue of $C(P')$. □

Theorem 2.6 gives us assurance that whenever l number of p_i's are equal to a real number p, $-p$ is an eigenvalue of $A(P)$ with multiplicity $l-1$. Let $P = \{p_1, p_2, \ldots, p_n\}$ be a non-decreasing sequence of positive real numbers. Suppose $p_1, p_2, \ldots, p_n$ takes s distinct values $q_1, q_2, \ldots, q_s$. To represent this situation, we rewrite P as $\{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$, where $k_1 + k_2 + \cdots + k_s = n$. Define an $s \times s$ matrix $C(P) = [c_{ij}]$ by

$$c_{ij} = \begin{cases} k_j q_j & \text{if } i \neq j, \\ (k_j - 1)q_j & \text{if } i = j. \end{cases}$$

Therefore, by using a similar approach to Theorem 2.8, we have the following result:

Theorem 2.9. *Let $P = \{p_1, p_2, \ldots, p_n\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ be a finite sequence of nonzero real numbers. Then the following are true:*

(a) *$-q_i$ is an eigenvalue of $A(P)$ with multiplicity $k_i - 1$.*

(b) *Every eigenvalue of $C(P)$ is also an eigenvalue of $A(P)$.*

(c) *$-q_i, 1 \leq i \leq s$ is not an eigenvalue of $C(P)$.*

An immediate consequence of Theorem 2.9 can be given as follows:

Corollary 2.10. *Let $P = \{p_1, p_2, \ldots, p_n\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ be a finite sequence of nonzero real numbers. Then*

$$\det C(P) = (-1)^{s-1}(n-1)q_1 q_2 \cdots q_s.$$

Proof. By Theorem 2.8, $-q_i$ is an eigenvalue of $A(P)$ with multiplicity $k_i - 1$ and $-q_i$, $1 \le i \le s$, is not an eigenvalue of $C(P)$. Therefore, the result follows from Theorem 2.1 as $\det C(P)$ equals to the product of eigenvalues of $C(P)$. □

Theorem 2.11. *Let $P = \{p_1, p_2, \ldots, p_n\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ be a finite sequence of nonzero real numbers and let $\Phi_{C(P)}(x) = x^n + c_1 x^{n-1} + \cdots + c_n$ be the characteristic polynomial of $C(P)$. Then*

$$c_r = -\sum_S \Big(\sum_{i \in S} k_i - 1\Big) \prod_{j \in S} q_j,$$

where the product runs over all subsets S of $\{1, 2 \ldots, n\}$ of order r.

Proof. The coefficient c_r of $\Phi_{C(P)}(x)$ is given by the following formula

$$c_r = (-1)^r (\textit{Sum of } r \times r \textit{ principal minors}).$$

Let $M(i_1, i_2 \ldots, i_r)$ be a principal minor obtained by taking i_j-th, $1 \le j \le r$ rows and corresponding columns. Then $M(i_1, i_2 \ldots, i_r) = C(P')$ where $P' = \{q_{i_1}^{k_{i_1}}, q_{i_2}^{k_{i_2}}, \ldots, q_{i_r}^{k_{i_r}}\}$. Now

$$\begin{aligned}
\det C(P') &= \begin{vmatrix} (k_{i_1}-1)q_{i_1} & k_{i_2}q_{i_2} & \cdots & k_{i_r}q_{i_r} \\ k_{i_1}q_{i_1} & (k_{i_2}-1)q_{i_2} & \cdots & k_{i_r}q_{i_r} \\ \vdots & \vdots & \cdots & \vdots \\ k_{i_1}q_{i_1} & k_{i_2}q_{i_2} & \cdots & (k_{i_r}-1)q_{i_r} \end{vmatrix} \\
&= q_{i_1}q_{i_2}\cdots q_{i_r} \begin{vmatrix} k_{i_1}-1 & k_{i_2} & \cdots & k_{i_r} \\ k_{i_1} & k_{i_2}-1 & \cdots & k_{i_r} \\ \vdots & \vdots & \cdots & \vdots \\ k_{i_1} & k_{i_2} & \cdots & k_{i_r}-1 \end{vmatrix} \\
&= (-1)^{r-1}(k_{i_1} + k_{i_2} + \cdots + k_{i_r} - 1) q_{i_1}q_{i_2}\cdots q_{i_r}.
\end{aligned}$$

Therefore, combining all possibilities, we get

$$c_r = -\sum_S \Big(\sum_{i \in S} k_i - 1\Big) \prod_{j \in S} q_j.$$

This completes the proof of the theorem. □

Example 2.12. Consider the finite sequence $P = \{2, 2, 2, 3, 3, 3\}$. Then

$$A(P) = \begin{bmatrix} 0 & 2 & 2 & 3 & 3 & 3 \\ 2 & 0 & 2 & 3 & 3 & 3 \\ 2 & 2 & 0 & 3 & 3 & 3 \\ 2 & 2 & 2 & 0 & 3 & 3 \\ 2 & 2 & 2 & 3 & 0 & 3 \\ 2 & 2 & 2 & 3 & 3 & 0 \end{bmatrix},$$

and

$$C(P) = \begin{bmatrix} 4 & 9 \\ 6 & 6 \end{bmatrix}.$$

In this case regular eigenvalues of $A(P)$ are -2 (multiplicity 2) and -3 (multiplicity 2). The non-regular eigenvalues are roots of the quadratic equation

$$x^2 - 10x - 30 = 0.$$

Therefore, eigenvalues of $A(P)$ are -2^2, -3^2, $5 \pm \sqrt{55}$.

2.1 Inverse and eigenvalue bounds

In this section, we first provide the inverse of a nonsingular multipart matrix. By corollary 2.2, if $p_i \neq 0$ for all $i = 1, 2, \ldots, n$, then $A(P)$ is nonsingular. For that case, we can find $A(P)^{-1}$ explicitly.

Theorem 2.13. *Let $P = \{p_1, p_2, \ldots, p_n\}$ be a finite sequence of nonzero real numbers. Then*

$$A(P)^{-1} = A(P')^T - \frac{n-2}{n-1} diag\Big[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\Big],$$

where $P' = \{\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\}$.

Proof. Let $D = diag[p_1, p_2, \ldots, p_n]$. Observe that, $A(P) = (J_n - I_n)D$, where J_n is the all-ones square matrix of order n. Then, we have

$$\frac{1}{n-1}\Big(J_n - (n-1)I_n\Big)\Big(J_n - I_n\Big) = \frac{1}{n-1}\Big(J_n^2 - (n-1)J_n - J_n + (n-1)I_n\Big) = I_n$$

and

$$\frac{1}{n-1}\Big(J_n - I_n\Big)\Big(J_n - (n-1)I_n\Big) = \frac{1}{n-1}\Big(J_n^2 - (n-1)J_n - J_n + (n-1)I_n\Big) = I_n.$$

Therefore, $\Big(J_n - I_n\Big)^{-1} = \frac{1}{n-1}\Big(J_n - (n-1)I_n\Big)$. Hence,

$$\begin{aligned} A(P)^{-1} = D^{-1}\Big(J_n - I_n\Big)^{-1} &= \frac{1}{n-1}D^{-1}\Big((J_n - I_n) - (n-2)I_n\Big) \\ &= \frac{1}{n-1}A(P')^T - \frac{n-2}{n-1}diag\Big[\frac{1}{p_1}, \frac{1}{p_2}, \ldots, \frac{1}{p_n}\Big]. \end{aligned}$$

This completes the proof of the theorem. □

Now we provide bounds for the eigenvalues for $A(P)$. For an $n \times n$ matrix with real eigenvalues, we arrange its eigenvalues as follows:

$$\lambda_n \leq \lambda_{n-1} \leq \cdots \lambda_2 \leq \lambda_1.$$

Theorem 2.14. *Let $P = \{p_1, p_2, \ldots, p_n\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ be a finite sequence of nonzero real numbers. Then the negative eigenvalues of $C(P)$ satisfy*

$$-q_s < \lambda_s < -q_{s-1} < \lambda_{s-1} < \cdots < -q_2 < \lambda_2 < -q_1.$$

Proof. Let $D = \operatorname{diag}[\sqrt{k_1 q_1}, \sqrt{k_2 q_2}, \ldots, \sqrt{k_s q_s}]$. Then $D^{\frac{1}{2}} C(P) D^{-\frac{1}{2}}$ is symmetric, and it has the same eigenvalues as $C(P)$. Let B be the $s \times s$ matrix whose each entry in the i-th column is $k_i q_i$. Then

$$D^{\frac{1}{2}} C(P) D^{-\frac{1}{2}} = D^{\frac{1}{2}} B D^{-\frac{1}{2}} + \operatorname{diag}[-q_1, -q_2, \ldots, -q_s].$$

Since the matrix $D^{\frac{1}{2}} B D^{-\frac{1}{2}}$ has eigenvalues 0 (with multiplicity $n-1$) and $\sum p_i$ (with multiplicity 1). Therefore, by Weyl's inequality [6, Theorem 4.3.1], we conclude that

$$-q_s < \lambda_s < -q_{s-1} < \lambda_{s-1} < \cdots < -q_2 < \lambda_2 < -q_1.$$

Now the eigenvalues of B are n (with multiplicity 1) and 0 (with multiplicity $s-1$). Hence the required inequalities follow, as $-q_i$ is not an eigenvalue of $C(P)$. □

By Theorem 2.14, we conclude that no two non-regular eigenvalues of a multipart matrix are equal. Let $P = \{p_1, p_2, \ldots, p_n\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ be a finite sequence of non-decreasing positive real numbers, and let $\mu_1 > \mu_2 > \cdots > \mu_s$ be the non-regular eigenvalues of $A(P)$. Then we can write

$$\begin{aligned}\lambda_1 = \mu_1 &= -\sum_{i=2}^{n} \lambda_i \\ &= \sum_{i=1}^{s} (k_i - 1) q_i + \sum_{i=2}^{s} -\mu_i.\end{aligned}$$

Now, by Theorem 2.14, $q_i < -\mu_i < q_{i-1}$ for $2 \leq i \leq s$. Therefore,

$$\sum p_i - p_n < \lambda_1 < \sum p_i - p_1.$$

In the next theorem, we provide a lower bound for the positive eigenvalue of a multipart matrix.

Theorem 2.15. *Let $P = \{p_1, p_2, \ldots, p_n\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ be a finite sequence of positive real numbers. Then*

$$\lambda_1 \geq \frac{1}{2} \sum_{i=1}^{s} (k_i - 1) q_i + \frac{s}{2} \sqrt[s]{(n-1) q_1 q_2 \cdots q_s}.$$

Proof. Since $\lambda_2 < 0 < \lambda_1$ and trace of $A(P)$ is 0, so $\lambda_1 = -(\lambda_2 + \lambda_3 + \cdots + \lambda_n)$. Let $\lambda_1 = \mu_1 > \mu_2 > \cdots > \mu_s$ be the non-regular eigenvalues of $A(P)$.

$$2|\lambda_1| = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{s} (k_i - 1) q_i + \sum_{i=1}^{s} |\mu_i|.$$

Now, applying A.M.$\geq$ G.M. on $|\mu_1|,\ |\mu_2|,\ \ldots,\ |\mu_s|$ and using Corollary 2.10, we get

$$|\lambda_1| \geq \frac{1}{2}\sum_{i=1}^{s}(k_i-1)q_i + \frac{s}{2}\sqrt[s]{(n-1)q_1q_2\cdots q_s}.$$

This completes the proof. $\square$

Remark 2.16. The bound provided in Theorem 2.14 is sharp. The equality holds if $P = \{p,p,p,\ldots,p\}$ or $P = \{p,q\}$.

In the next theorem, we provide an upper bound for the largest eigenvalue (spectral radius) of multipart matrix.

Theorem 2.17. *Let $P = \{p_1,p_2,\ldots,p_n\} = \{q_1^{k_1},q_2^{k_2},\ldots,q_s^{k_s}\}$ be a finite sequence of nonzero real numbers. Then*

$$\sigma(A(P)) = \lambda_1 \leq (n-1)(p_1p_2\ldots p_n)^{\frac{n-1}{n}}.$$

Proof. By Theorem 2.14, λ_1 is the only positive eigenvalue $A(P)$. Therefore $\frac{1}{\lambda_1}$ is the largest eigenvalue of $A(P)^{-1}$. Then for any nonzero vector $X = [x_1\ x_2\ \cdots\ x_n]^T \in \mathbb{R}^n$, we have

$$\frac{1}{\lambda_1} \geq \frac{X^TA(P)^{-1}X}{X^TX}.$$

We take X to be the column vector with all component equal to 1. Then we have

$$\begin{aligned}\frac{1}{\lambda_1} &\geq \frac{1}{n}\Big[\text{sum of entries of of } A(P)^{-1}\Big]\\ &= \frac{1}{n(n-1)}\sum\frac{1}{p_i}\\ &= \frac{p_1+p_2+\cdots+p_n}{n(n-1)p_1p_2\ldots p_n}.\end{aligned}$$

Applying A.M.$\geq$G.M. on $p_1,p_2,\ldots,p_n$ we get

$$\frac{1}{\lambda_1} \geq \frac{1}{(n-1)(p_1p_2\ldots p_n)^{\frac{n-1}{n}}}.$$

Therefore, $\lambda_1 \leq (n-1)(p_1p_2\ldots p_n)^{\frac{n-1}{n}}$. $\square$

Remark 2.18. The bound provided in the above theorem is sharp. The equality holds if $P = \{p,p,p,\ldots,p\}$ or $P = \{p,q\}$.

3 Applications in spectral graph theory

Now we provide some applications of the multipart matrices. We have already observed that the quotient matrix of the adjacency matrix of a complete multipartite graph is a multipart matrix. Here we consider two connectivity matrices of a complete multipartite graph, namely, the adjacency matrix and the normalized adjacency matrix.

3.1 Adjacency eigenvalues of complete multipartite graphs

In this subsection, we try to analyze the spectral properties of the adjacency matrix of complete multipartite graphs. The complete multipartite graph $K_{p_1,p_2,\ldots,p_k}$ on n vertices has nullity $n-k$. An orthogonal set of eigenvectors corresponding to 0 can be constructed as follows:

Note that any nonzero vector $X = [x_1\ x_2\ \cdots\ x_n]^T \in \mathbb{R}^n$, whose entries satisfy the equation $\sum_{j\sim i} x_j = 0$, for all $1 \le i \le n$, is an eigenvector corresponding to 0. Let O_r denote the r-component zero row-vector, and for $i > 1$ define the row-vectors X^i_j by

$$X^i_j = \boldsymbol{e}_1(i)^T + \boldsymbol{e}_2(i)^T + \cdots + \boldsymbol{e}_j(i)^T - j\boldsymbol{e}_{j+1}(i)^T, \quad \textit{for all } 1 \le j \le i-1,$$

where $\boldsymbol{e}_j(i)$ is the j-th standard basis element of $\mathbb{R}^i$. Now, for $P_i \ge 2$, we define

$$Y_i(j) = [O_{p_1}\ O_{p_2}\ \cdots\ O_{p_{i-1}}\ X^{p_i}_j\ O_{p_{i+1}}\ \cdots\ O_{p_k}]^T, \quad \textit{for all } 1 \le i \le k, 1 \le j \le p_i - 1.$$

Then, for each $p_i > 1$, the set $\{Y_i(j) | 1 \le i \le k, 1 \le j \le p_i - 1 \text{ and } p_i > 1\}$ contains $n-k$ orthogonal eigenvectors corresponding to 0. Therefore, a complete multipartite graph $K_{p_1,p_2,\ldots,p_k}$ on n vertices has at most k distinct nonzero eigenvalues. Again, the nonzero eigenvalues of the adjacency matrix are eigenvalues of the multipart matrix $A(P)$, where $P = \{p_1, p_2, \cdots, p_k\}$.

Let $P = \{p_1, p_2, \ldots, p_k\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ with $\sum k_i = k$ and $\sum k_i q_i = \sum p_i = n$. Then Theorem 2.6, Theorem 2.9, and Theorem 2.14 provide alternative proofs for some of the existing results [3, 4] related to adjacency eigenvalues of $K_{p_1,p_2,\ldots,p_k}$.

Theorem 3.1. *[4] Let $P = \{p_1, p_2, \ldots, p_k\} = \{q_1^{k_1}, q_2^{k_2}, \ldots, q_s^{k_s}\}$ be non-decreasing sequence of natural numbers with $p_1 + p_2 + \cdots + p_k = n$. If λ_1 is the positive adjacency eigenvalue of $K_{p_1,p_2,\ldots,p_k}$, then the following are true:*

(a) *0 is an eigenvalue with multiplicity $n-k$.*

(b) *$-q_i$ is an eigenvalue with multiplicity $k_i - 1$.*

(c) *If $\mu_s < \mu_{s-1} < \cdots < \mu_1$ are the non-regular eigenvalues, then*

$$-q_s < \mu_s < -q_{s-1} < \mu_{s-1} < \cdots \le -q_2 < \mu_2 \le -q_1.$$

Theorem 3.2. *[3] The characteristics polynomial of $K_{p_1,p_2,\cdots,p_k}$ is given by*

$$x^{n-k}(x^k - c_2x^{k-2} - c_3x^{k-3} - c_4x^{k-4} - \cdots - c_k),$$

where $c_i = (i-1)\sum_S \prod_{j\in S} p_j$, $2 \le i \le k$ and the summation runs over all subsets S of $\{1, 2\ldots, k\}$ of order i.

Now since all non-regular eigenvalues of $A(P)$ are also eigenvalues of $C(P)$, we have the following characteristic polynomial formula:

Theorem 3.3. *Let $P = \{p_1, p_2, \dots, p_k\} = \{q_1^{k_1}, q_2^{k_2}, \dots, q_s^{k_s}\}$ be non-decreasing sequence of natural numbers with $p_1 + p_2 + \cdots + p_k = n$. Then the characteristics polynomial of $K_{p_1,p_2,\cdots,p_k}$ is given by*

$$x^{n-k}\Big(\prod_{j=1}^{s}(x+q_j)^{k_j-1}\Big)(x^s - c_1x^{s-1} - c_2x^{s-2} - c_3x^{s-3} - \cdots - c_s),$$

where $c_r = \sum_S \left(\sum_{i\in S} k_i - 1\right)\prod_{j\in S} q_j$, $1 \le r \le s$ and the summation runs over all subsets S of $\{1, 2 \dots, k\}$ of order r.

Proof. The result follows from Theorem 3.1 and Theorem 2.11. $\square$

Example 3.4. Consider the complete multipartite graph $G = K_{2,2,2,3,3,3,4,4,4}$. The coefficients in the Theorem 3.3 are $c_1 = 18$, $c_2 = 130$, $c_3 = 192$. Consequently the characteristic polynomial of G is

$$\Phi_A(x) = x^{18}(x+2)^2(x+3)^2(x+4)^2(x^3 - 18x^2 - 130x - 192).$$

Therefore, the adjacency eigenvalues of $K_{2,2,2,3,3,3,4,4,4}$ are -4^2, -3.4884, -3^2, -2.3125, -2^2 0^{18}, 23.8009.

The next result gives a lower bound and an upper bound for the positive adjacency eigenvalue of complete multipartite graphs. The result is the immediate consequences of Theorem 2.15 and Theorem 2.17.

Theorem 3.5. *Let $P = \{p_1, p_2, \dots, p_k\} = \{q_1^{k_1}, q_2^{k_2}, \dots, q_s^{k_s}\}$ be non-decreasing sequence of natural numbers with $p_1 + p_2 + \cdots + p_k = n$. If λ_1 is the positive adjacency eigenvalue of $K_{p_1,p_2,\dots,p_k}$, then*

$$(k-1)(p_1p_2\dots p_k)^{\frac{k-1}{k}} \ge \lambda_1 \ge \frac{n}{2} - \frac{1}{2}\sum_{i=1}^{s} q_i + \frac{s}{2}\sqrt[s]{(k-1)q_1q_2\cdots q_s}.$$

Proof. Since $k_1q_1 + k_2q_2 + \cdots + k_sq_s = p_1 + p_2 + \cdots + p_k = n$. Hence the result follows from Theorem 2.15 and Theorem 2.17. $\square$

Remark 3.6. The equality for the bound in Theorem 3.5 holds if G is a regular complete multiprtite graph or a complete bipartite graph.

3.2 Normalized adjacency eigenvalues of complete multipartite graphs

Let $V = V_1 \cup V_2 \cup \cdots \cup V_k$ be the equitable partition for the vertex set of $G = K_{p_1,p_2,\dots,p_k}$ consisting of independent subsets of G. Then the quotient matrix for $\mathcal{A}$ is given by

$$\mathcal{Q} = D^{-1}Q,$$

where Q is the quotient matrix for A and $D = diag[n-p_1, n-p_2, \dots, n-p_k]$. The matrix $\mathcal{Q}$ is similar to the matrix QD^{-1}. We observe that the matrix QD^{-1} is a multipart matrix for the finite sequence $\{\frac{p_1}{n-p_1}, \frac{p_2}{n-p_2}, \dots, \frac{p_k}{n-p_k}\}$. Note that any eigenvalue of $\mathcal{Q}$ is also an eigenvalue of $\mathcal{A}$. Thus, as nullity of $\mathcal{A}$ is $n-k$, all nonzero eigenvalues of $\mathcal{A}$ is also eigenvalues of the multipart matrix QD^{-1}. Now we have the following result:

Theorem 3.7. *Let $P = \{p_1, p_2, \dots, p_k\} = \{q_1^{k_1}, q_2^{k_2}, \dots, q_s^{k_s}\}$ be non-decreasing sequence of natural numbers with $p_1+p_2+\cdots+p_k = n$. Then the eigenvalues of the normalized adjacency matrix $\mathcal{A}$ of $K_{p_1,p_2,\dots,p_k}$ has the following properties:*

(a) *0 is an eigenvalue with multiplicity $n-k$ and 1 is a simple eigenvalue.*

(b) $-\dfrac{q_i}{n-q_i}$ *is a eigenvalue of $\mathcal{A}$ with multiplicity $k_i - 1$.*

(c) *All non-regular eigenvalues are distinct.*

(d) *If $1 = \mu_1 > \mu_2 > \cdots > \mu_s$ are the non-regular eigenvalues of $\mathcal{A}$, then*

$$\frac{-q_s}{n-q_s} > \mu_s > \frac{-q_{s-1}}{n-q_{s-1}} > \mu_{s-1} > \frac{-q_{s-2}}{n-q_{s-2}} > \cdots \frac{-q_2}{n-q_2} > \mu_2 > \frac{-q_1}{n-q_1}.$$

Proof. Since $\mathcal{A}$ is a row stochastic matrix, 1 is an eigenvalue of $\mathcal{A}$. Note that all the nonzero eigenvalues of $\mathcal{A}$ are eigenvalues of $A(\mathcal{P})$, where

$$\mathcal{P} = \Big\{\frac{p_1}{n-p_1}, \frac{p_2}{n-p_2}, \dots, \frac{p_k}{n-p_k}\Big\} = \Big\{\Big(\frac{q_1}{n-q_1}\Big)^{k_1}, \Big(\frac{q_2}{n-q2}\Big)^{k_2} \dots, \Big(\frac{q_s}{n-q_s}\Big)^{k_s}\Big\}.$$

Therefore, the result follows from Theorem 2.6, Theorem 2.9, Theorem 2.14, and Sylvestar law of inertia. □

Example 3.8 (Biregular complete multipartite graphs)**.** Let $K_{p_1,p_2,\dots,p_k}$ be a biregular complete multipartite graph. Then $P = \{p^{k_1}, q^{k_2}\}$. In that case, the normalized adjacency eigenvalues are 0 (with multiplicity $n-k$), $-\dfrac{p}{n-p}$ (with multiplicity $k_1 - 1$), $-\dfrac{q}{n-q}$ (with multiplicity $k_2 - 1$), $-\dfrac{(k-1)pq}{(n-p)(n-q)}$ (with multiplicity 1), and 1 (with multiplicity 1).

Example 3.9 (Triregular complete multipartite graphs)**.** Let $K_{p_1,p_2,\dots,p_k}$ be a triregular complete multipartite graph. Then $P = \{p^{k_1}, q^{k_2}, r^{k_3}\}$. In that case, the normalized adjacency eigenvalues are 0 (with multiplicity $n-k$), $-\dfrac{p}{n-p}$ (with multiplicity $k_1 - 1$), $-\dfrac{q}{n-q}$ (with multiplicity $k_2 - 1$), $-\dfrac{r}{n-r}$ (with multiplicity $k_3 - 1$), 1 (with multiplicity 1), and two negative non-regular eigenvalues are roots of the equation

$$x^2 + \Big(1 - \frac{(k_1-1)p}{n-p} - \frac{(k_2-1)q}{n-q} - \frac{(k_3-1)r}{n-r}\Big)x + \frac{2pqr}{(n-p)(n-q)(n-r)}.$$

In particular, if $k_1 = k_2 = k_3 = 1$, i.e., $G = k_{p,q,r}$, then its normalized eigenvalues are 0^{n-3}, 1, $-\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{8pqr}{(n-p)(n-q)(n-r)}}$.

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