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On the structure of the fundamental subspaces of acyclic matrices with 0 in the diagonal

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Abstract

A matrix is called acyclic if replacing the diagonal entries with 0, and the nonzero diagonal entries with 1, yields the adjacency matrix of a forest. In this paper we show that the null space and the rank of an acyclic matrix with 0 in the diagonal is obtained from the null space and the rank of the adjacency matrix of the forest by multipliying by nonsingular diagonal matrices. We combine these with an algorithm for finding a sparsest basis of the null space of a forest to provide an optimal time algorithm for finding a sparsest basis of the null space of acyclic matrices with 0 in the diagonal.

1 Introduction

Throughout this article, all graphs are assumed to be finite, undirected, and without loops and multiple edges. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. The $(i,j)-$entry of a matrix M is denoted $M_{i,j}$. Unless specified for a particular example, we assume our matrices are defined over an arbitrary field $\mathbb{F}$. Following the notation in [10], we denote by $\mathcal{M}_{\mathbb{F}}(G)$ the set of all matrices M over $\mathbb{F}$ with rows and columns indexed by $V(G)$, so that for every two distinct vertices $u, v \in V(G)$, the $M_{u,v} \neq 0$ if and only if $\{u, v\} \in E(G)$. Notice that the diagonal entries are allowed to be nonzero. A matrix M over $\mathbb{F}$ is said to be *acyclic* if $M \in \mathcal{M}_{\mathbb{F}}(F)$ for a forest F. If the forest F is a tree, then M is said to be *tree-patterned*. See Figure 1 for an example.

Given a graph G, the adjacency matrix of G, denoted by $A(G)$, is the only $(0,1)-$matrix

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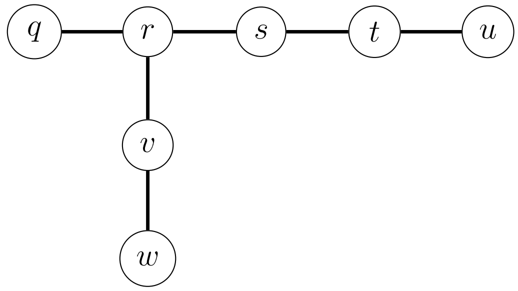

$$M = \begin{pmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & -1 & 0 & 0 & 5 & 0 \\ 0 & -2 & 0 & -7 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 \end{pmatrix}$$

T

Figure 1: A tree T and a matrix $M \in \mathcal{M}_{\mathbb{R}}(T)$, where the labels of the vertices are in alphabetical order (i.e., vertex q correspond to the first row and column of M, vertex r to the second, and so on).

in $\mathcal{M}_{\mathbb{F}}(G)$ with zero diagonal. For T in Figure 1 we have

$$A(T) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

where rows and columns are in alphabetical order (i.e., the first row and column correspond to vertex q, the second to r, and so on).

In [10], the following lemma was proved.

Lemma 1.1. *Let M be an acyclic matrix over a field $\mathbb{F}$. Then there exist a finite-dimensional field extension $\mathbb{E}$ of $\mathbb{F}$ and a diagonal matrix D over $\mathbb{E}$ such that $D^{-1}MD$ is symmetric.*

Due to a previous version of Lemma 1.1 (which first appeared in [13]), most of the study of acyclic matrices was done on symmetric matrices. The matrices considered in this article do not need to be symmetric. This is done because the proofs work almost identically, and in this way there is no need to calculate the necessary diagonal matrix over the field extension.

The fundamental spaces of a matrix M are its null space, $\mathcal{N}\mathit{ull}(M)$, and its rank, $\mathcal{R}\mathit{ank}(M)$. The structure of the fundamental spaces of graph-patterned matrices has been studied in depth for symmetric tree-patterned matrices allowing nonzero entries in the diagonal, see for instance [3, 11, 12, 14]. While most of the focus on these articles has been on the dimension of the null space, the important problem of finding a basis for it has not been studied.

Given $M \in \mathcal{M}_{\mathbb{F}}(G)$, the *null support* of M, denoted $\mathcal{S}\mathit{upp}(M)$, is the set of vertices of G that have nonzero entries in at least one vector from $\mathcal{N}\mathit{ull}(M)$. In other words, $v \in \mathcal{S}\mathit{upp}(M)$ if there is a vector $\vec{x} \in \mathcal{N}\mathit{ull}(M)$ with $x_v \neq 0$, where x_v is the coordinate of $\vec{x}$ corresponding to vertex v. For the adjacency matrix $A(T)$ of the tree T in Figure 1, $\mathcal{N}\mathit{ull}(A(T))$ is generated by $(1,0,-1,0,1,0,0)^T$, and thus we have $\mathcal{Supp}(A(T)) = \{q,s,u\}$. Similarly, $\mathcal{Null}(M)$, for M in Figure 1, is generated by $(1,0,3,0,-9/5,0,0)^T$, and $\mathcal{Supp}(M) = \{q,s,u\}$. Notice that in this case $\mathcal{Supp}(M)$ and $\mathcal{Supp}(A(T))$ coincide. In Section 2 we show that this is the case for every tree (and for every forest). In [6] the null support of adjacency matrices of forests has been studied in depth. The authors provided a decomposition for any forest into an S-forest (a forest that has a unique maximum independent set), and an N-forest (a forest that has a unique maximum matching). They showed that all the information of the null space of $A(F)$ can be obtained from the S-forests and N-forests related to F. It was implicitly shown that $\mathcal{Null}(A(F))$ coincides with the intersection of all the maximum independent sets of F. In [7] the authors studied *sparsest* (i.e., has the fewest nonzeros) $\{-1,0,1\}$ basis for the null space of a forest, and provided an algorithm that finds such a basis in optimal time (i.e., in linear time with respect to the number of nonzero entries of the basis). The problem of finding a sparsest basis of the null space of a matrix is important for numerical applications. Finding such a basis in general is known to be NP-complete [2] and even hard to approximate [5].

Let $\mathcal{M}_{\mathbb{F},0}(G)$ be the set of matrices in $\mathcal{M}_{\mathbb{F}}(G)$ with zero in the diagonal. In Section 2 we show that given a forest F, the null space of any matrix in $\mathcal{M}_{\mathbb{F},0}(F)$ can be obtained by multiplying the null space of F by a suitable nonsingular diagonal matrix. This allows the use of the tools developed in [6, 7] for the study of the null space of said matrices. In particular, we use the results of [7] to give an optimal time algorithm for finding a sparsest basis for the null space of the matrix. In Section 3 we prove that the rank of any matrix in $\mathcal{M}_{\mathbb{F},0}(F)$ can be found by multiplying the rank of F by a suitable nonsingular diagonal matrix.

Square matrices with zero diagonal are also known as hollow matrices. They are usual in chemistry (see [9]), via the notion of distance matrix. Grood et al, see [4], studied the minimum possible rank among all the symmetric matrices with zero diagonal.

2 On the null space

The null space of a graph is the direct sum of the null spaces of its connected components. In a similar fashion, the null space of a matrix $M \in \mathcal{M}_{\mathbb{F}}(G)$ is the direct sum of the null spaces of M over the connected components of G. Because of this, we restrict our study the null space of matrices over trees.

Lemma 2.1, which is fundamental for our results, first appeared in [10] as Theorem 8(i).

Lemma 2.1. *[10] Let F be a forest, and $M \in \mathcal{M}_{\mathbb{F},0}(F)$. If $\{v,w\} \in \mathcal{Supp}(M)$, then $\{v,w\} \notin E(F)$.*

Let T be a tree, $M \in \mathcal{M}_{\mathbb{F},0}(T)$, and v be a vertex of T. For each vertex $w \in T$, let vPw be the unique directed path from v to w in T. In this sense vPw and wPv are different, because we care about the direction. We define the *v-scalation of M* as the nonsingular diagonal matrix given by

$$D^{(M,v)}_{w_1,w_2} = \begin{cases} \prod_{\substack{u\in \mathcal{Supp}(M),\\ (u,t)\in vPw_1}} M^{-1}_{t,u} \prod_{\substack{u\in \mathcal{Supp}(M),\\ (t,u)\in vPw_1}} M_{t,u} & \text{if } w_1 = w_2,\\ 0 & \text{otherwise.}\end{cases}$$

For matrix M in Figure 1, we have

$$\begin{aligned}
D^{(M,q)}_{q,q} &= 1\\
D^{(M,q)}_{r,r} &= \frac{1}{3}\\
D^{(M,q)}_{s,s} &= \frac{1}{3}(-1) = -\frac{1}{3}\\
D^{(M,q)}_{t,t} &= \frac{1}{3}(-1)\frac{1}{3} = -\frac{1}{9}\\
D^{(M,q)}_{u,u} &= \frac{1}{3}(-1)\frac{1}{3}5 = -\frac{5}{9}\\
D^{(M,q)}_{v,v} &= \frac{1}{3}\\
D^{(M,q)}_{w,w} &= \frac{1}{3},
\end{aligned}$$

thus, the q-scalation is

$$D^{(M,q)} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{1}{3} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{9} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -\frac{5}{9} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \end{pmatrix}.$$

Similarly, the s-scalation is

$$D^{(M,s)} = \begin{pmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{3} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{5}{3} & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and the u-scalation is

$$D^{(M,u)} = \begin{pmatrix} -\frac{9}{5} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -\frac{3}{5} & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & \frac{3}{5} & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{5} & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -\frac{3}{5} & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{5} \end{pmatrix}.$$

Notice that

$$D^{(M,q)}\begin{pmatrix}1\\0\\3\\0\\-\frac{9}{5}\\0\\0\end{pmatrix}=\begin{pmatrix}1\\0\\-1\\0\\1\\0\\0\end{pmatrix}$$

$$D^{(M,s)}\begin{pmatrix}1\\0\\3\\0\\-\frac{9}{5}\\0\\0\end{pmatrix}=\begin{pmatrix}-3\\0\\3\\0\\-3\\0\\0\end{pmatrix}=-3\begin{pmatrix}1\\0\\-1\\0\\1\\0\\0\end{pmatrix}$$

$$D^{(M,u)}\begin{pmatrix}1\\0\\3\\0\\-\frac{9}{5}\\0\\0\end{pmatrix}=\begin{pmatrix}-\frac{9}{5}\\0\\\frac{9}{5}\\0\\-\frac{9}{5}\\0\\0\end{pmatrix}=-\frac{9}{5}\begin{pmatrix}1\\0\\-1\\0\\1\\0\\0\end{pmatrix}.$$

What we are seeing is that in our example $D^{(M,a)}\vec{x}$ is in $\mathcal{Null}(A(T))$ for every $a \in \mathcal{Supp}(M)$ and every $\vec{x} \in \mathcal{Null}(M)$. We are going to show that this is the case in general, but first we present a useful lemma, that follows from the definition of $D^{(M,v)}$ and Lemma 2.1.

Lemma 2.2. *Let T be a tree, $M \in \mathcal{M}_{\mathbb{F},0}(F)$, $u, v \in V(T)$, $vPu = (v = v_0, v_1, \ldots, u_1, u)$ and $vPu_2 = (v = v_0, v_1, \ldots, u_1, u, u_2)$ two directed paths in T. The following statements are true.*

1. *If $u_1, u_2 \in \mathcal{Supp}(M)$,*
$$D^{(M,v)}_{u_2,u_2} = D^{(M,v)}_{u_1,u_1} M^{-1}_{u,u_1} M_{u,u_2}.$$
2. *If $u_1 \in \mathcal{Supp}(M)$, $u_2 \notin \mathcal{Supp}(M)$,*
$$D^{(M,v)}_{u_2,u_2} = D^{(M,v)}_{u_1,u_1} M^{-1}_{u,u_1}.$$
3. *If $u_1 \notin \mathcal{Supp}(M)$, $u_2 \in \mathcal{Supp}(M)$,*
$$D^{(M,v)}_{u_2,u_2} = D^{(M,v)}_{u_1,u_1} M_{u,u_2}.$$
4. *If $u_1, u, u_2 \notin \mathcal{Supp}(M)$,*
$$D^{(M,v)}_{u_2,u_2} = D^{(M,v)}_{u_1,u_1}.$$

5. *If $u \in \mathcal{S}upp(M)$,*
$$D^{(M,v)}_{u_2,u_2} = D^{(M,v)}_{u_1,u_1} M_{u_1,u} M^{-1}_{u_2,u}.$$

As a direct consequence of Lemma 2.2, the matrices $D^{(M,v)}$ and $\left(D^{(M,v)}\right)^{-1}$ can be obtained in linear time over the number of vertices, as at most one multiplication must be done at each vertex.

Theorem 2.3. *Given a tree T, a matrix $M \in \mathcal{M}_{\mathbb{F},0}(T)$ and a vertex $v \in \mathcal{S}upp(M)$, a vector $\vec{x}$ is in $\mathcal{N}ull(M)$ if and only if $D^{(M,v)}\vec{x}$ is in $\mathcal{N}ull(A(T))$.*

Proof. Suppose $\vec{x} = (x_1, \ldots, x_n) \in \mathcal{N}ull(M)$, and let $v \in \mathcal{S}upp(M)$.

We want to show $A(T)D^{(M,v)}\vec{x} = 0$. Consider $\left(A(T)D^{(M,v)}\vec{x}\right)_u$. If $\vec{x}_w = 0$ for all $w \sim u$, then

$$\left(A(T)D^{(M,v)}\vec{x}\right)_u = \sum_{w \sim u} D^{(M,v)}_{w,w} x_w = 0.$$

Otherwise, let $(w_1, u) \in vPu$. If $x_{w_1} \neq 0$, then applying Lemma 2.2 we get:

$$\begin{aligned}
(A(T)D\vec{x})_u &= D_{w_1,w_1} x_{w_1} + \sum_{\substack{w \sim u,\\ w \neq w_1}} D_{w,w} x_w \\
&= D_{w_1,w_1} x_{w_1} + \sum_{\substack{w \sim u,\\ w \neq w_1}} x_w D_{w_1,w_1} \left(M^{-1}_{u,w_1} M_{u,w}\right) \\
&= D_{w_1,w_1} M^{-1}_{u,w_1} \left(M_{u,w_1} x_{w_1} + \sum_{\substack{w \sim u,\\ w \neq w_1}} x_w M_{u,w} \right) \\
&= D_{w_1,w_1} M^{-1}_{u,w_1} (M\vec{x})_u \\
&= 0.
\end{aligned}$$

If $x_{w_1} = 0$, and $x_w \neq 0$ for some $w \sim u$, Lemma 2.2 implies:

$$\begin{aligned}
(A(T)D\vec{x})_u &= \sum_{\substack{w \sim u,\\ w \neq w_1}} D_{w,w} x_w \\
&= \sum_{\substack{w \sim u,\\ w \neq w_1}} x_w D_{w_1,w_1} M_{u,w} \\
&= D_{w_1,w_1} (M\vec{x})_u \\
&= 0
\end{aligned}$$

Therefore $A(T)D^{(M,v)}\vec{x} = 0$.

For the reciprocal, similar arguments working with D^{-1} instead of D yield the result. $\square$

If we consider a forest instead of a tree, then a diagonal matrix can be obtained by choosing a vertex in each connected component, and working in a similar fashion. Let F be a forest, and $U \subset V(F)$ such that U has at most one vertex in each connected component of F. We define the *U-scalation of M* as the nonsingular diagonal matrix with

$$D^{(M,U)}_{w,w} = \prod_{v \in U} D^{(M,v)}_{w,w},$$

where $D^{(M,v)}_{w,w} = 1$ if v and w belong to different connected components of F.

In order to generalize Theorem 2.3 we need a set U with elements in every component with a vertex in $\mathcal{S}upp(M)$. Given a forest F with connected components $T_1, ..., T_k$, a set $U \subset \mathcal{S}upp(M)$ is *supp-transversal of M* if $U \cap V(T_i) \neq \emptyset$ whenever $\mathcal{S}upp(M) \cap V(T_i) \neq \emptyset$. We have the following.

Corollary 2.4. *Given a forest F and a matrix $M \in \mathcal{M}_{\mathbb{F},0}(F)$, a vector $\vec{x}$ is in $\mathcal{N}ull(M)$ if and only if $D^{(M,U)}\vec{x}$ is in $\mathcal{N}ull(A(F))$ for every set U supp-transversal of M.*

Corollary 2.5. *Let F be a forest, and $M, N \in \mathcal{M}_{\mathbb{F},0}(F)$. For every set U_1 supp-transversal of M and every set U_2 supp-transversal of N, $\vec{x} \in \mathcal{N}ull(M)$ if and only if $D^{(M,U_1)}(D^{(N,U_2)})^{-1}\vec{x}$ is in the null space of N.*

Corollary 2.6. *Given a forest F, and a pair of matrices $M, N \in \mathcal{M}_{\mathbb{F},0}(F)$, a vertex v is in $\mathcal{S}upp(M)$ if and only if v it is in $\mathcal{S}upp(N)$.*

Proof. If $v \in \mathcal{S}upp(M)$, there is some $\vec{x} \in \mathcal{N}ull(M)$ with $\vec{x}_v \neq 0$. Thus, by Corollary 2.5, the vector $D^{(M,v_1)}(D^{(N,v_2)})^{-1}\vec{x}$ is in $\mathcal{N}ull(N)$ for some $v_1 \in \mathcal{S}upp(M)$, $v_2 \in \mathcal{S}upp(N)$. But $\left(D^{(M,v_1)}(D^{(N,v_2)})^{-1}\vec{x}_v\right) \neq 0$ because $D^{(M,v_1)}$ and $D^{(N,v_2)}$ are nonsingular diagonal matrices. Therefore, $v \in \mathcal{S}upp(N)$. $\square$

Corollary 2.6 implies that the null support of matrices in $\mathcal{M}_{\mathbb{F},0}(F)$ depend only on F. Thus, we can talk about the null support of a forest F, $\mathcal{S}upp(F)$, as the null support of matrices in $\mathcal{M}_{\mathbb{F},0}(F)$.

The next two corollaries are given to illustrate the strength of Corollary 2.5, and the relation between the structure of a forest F and the null space of the matrices in $\mathcal{M}_{\mathbb{F},0}(F)$. In [6], the concept of the S-set of a tree T was introduced. It is the subgraph induced by $\mathcal{S}upp(T) \cup N(\mathcal{S}upp(T))$ (where $N(\mathcal{S}upp(T))$ denotes the neighborhood of $\mathcal{S}upp(T)$) and is denoted $\mathcal{F}_S(T)$. In other words, $\mathcal{F}_S(T)$ is the subgraph induced by the vertices in the null support and the neighbors of the vertices in the null support. One of the main results from [6] is the fact that the null space of a tree T is the same as the null space of $\mathcal{F}_S(T)$, extended with 0 to match the dimensions. The same holds true for matrices in $\mathcal{M}_{\mathbb{F},0}(T)$. And, using direct sum, the same holds true for forests.

In order to relate the null space of $\mathcal{F}_S(F)$ to the null space of a matrix $M \in \mathcal{M}_{\mathbb{F},0}(F)$, we introduce the following notation. Given a matrix $M \in \mathcal{M}_{\mathbb{F},0}(F)$, and G an induced subgraph of F, we denote by $M[G]$ the matrix obtained by deleting the rows and columns of vertices not in G. We do the same for vectors, $\vec{x}[G]$ denotes the vector obtained from $\vec{x}$ by deleting the coordinates corresponding to vertices not in G.

Corollary 2.7. *Let F be a forest, $M \in \mathcal{M}_{\mathbb{F},0}(F)$ and $\vec{x} \in \mathbb{F}^F$. Then $\vec{x} \in \mathcal{N}ull(M)$ if and only if:*

- $\vec{x}\,[\mathcal{F}_S(F)] \in \mathcal{N}ull(M\,[\mathcal{F}_S(F)])$, *and*
- $\vec{x}\,[V(F) \setminus \mathcal{F}_S(F)] = \vec{0}$.

A helpful result, implicit in [6], is the fact that $\mathcal{S}upp(T)$ is the intersection of all maximum independent sets of T. Which yields the following.

Corollary 2.8. *Let F be a forest, $M \in \mathcal{M}_{\mathbb{F},0}(F)$, and $v \in V(F)$. Then $v \in \mathcal{S}upp(M)$ if and only if v is in every maximum independent set of F.*

The next corollary, originally proved in [10], follows directly from Corollary 2.5 and the fact that the dimension of the rank of a tree is twice its matching number (see [1]).

Corollary 2.9. *If F is a forest and $M \in \mathcal{M}_{\mathbb{F},0}(F)$, then $\dim \mathcal{R}ank(M) = 2m$, where m is the size of a maximum matching in F.*

We can use the relation between the null space of a forest F and the null space of any matrix $M \in \mathcal{M}_{\mathbb{F},0}(F)$ to find a basis for the null space of M. This is done in Algorithm 1. Finding the forest F given the matrix M takes linear time, because it can be obtained by replacing the entries by 1, and the matrix has at most $2(n-1)$ nonzero entries (the edges of the forest). As $D^{(M,U)}$ does not change the support of a vector, a sparsest basis for the null space of F provides a sparsest basis for the null space of M once it is multiplied by $(D^{(M,U)})^{-1}$. In [7] the support of a forest was found in linear time with respect to the number of vertices of the forest, and, once the support was found, a $\{-1,0,1\}$ and sparsest basis for the null space of a forest was obtained in linear time with respect to the number of nonzero entries of the basis.

Using the support, finding $D^{(M,U)}$ and $(D^{(M,U)})^{-1}$ takes linear time on the number of vertices, as for each vertex only one operation needs to be done. Afterwards, multiplying the elements of the basis found using the algorithm from [7] by $(D^{(M,U)}_{v,v})^{-1}$ takes one operation per each nonzero entry in the vectors of the basis of the forest. Hence a sparsest basis for the null space of M can be found in linear time with respect to the maximum between the number of vertices and the number of nonzero entries in the vectors of a sparsest basis. We call such time *optimal*, as just writing a basis down takes at least linear time on the number of nonzero entries it has.

Algorithm 1: for finding a sparsest basis of the null space of an acyclic matrix with 0 in the diagonal.

1. INPUT: M, an acyclic matrix with 0 in the diagonal.

2. Find F such that $M \in \mathcal{M}_{\mathbb{F},0}(F)$.

3. Apply the algorithms from [7] to find a sparsest basis, $\mathcal{B}_F$ of $A(F)$ and $\mathcal{Supp}(F)$.

4. Find the connected components of F.

5. For each T_i connected component of F with $\mathcal{Supp}(A(T_i)) \neq \emptyset$ chose $v_i \in \mathcal{Supp}(A(T_i))$, and let U be the set of the chosen v_i.

6. Calculate $(D^{(M,U)})^{-1}$. To do this, root each T_i at v_i, and proooceed inductively as follows. First, let $(D^{(M,U)})^{-1}_{v_i,v_i} = 1$. Next, take a vertex $w \in T_i$ such that $(D^{(M,U)})^{-1}_{p(w),p(w)}$ has been defined, where $p(w)$ be is the parent of w. If $p(w) \in \mathcal{Supp}(A(T_i))$, let $(D^{(M,U)})^{-1}_{w,w} = M_{w,p(w)}(D^{(M,U)})^{-1}_{p(w),p(w)}$. If $p(w) \notin \mathcal{Supp}(A(T_i))$ and $w \in \mathcal{Supp}(A(T_i))$, let $(D^{(M,U)})^{-1}_{w,w} = M^{-1}_{p(w),w}(D^{(M,U)})^{-1}_{p(w),p(w)}$. If $w, p(w) \notin \mathcal{Supp}(A(T_i))$, let $(D^{(M,U)})^{-1}_{w,w} = M^{-1}_{p(w),w}(D^{(M,U)})^{-1}_{p(w),p(w)}$. Once this process stops, if $(D^{(M,U)})^{-1}_{w,w}$ was not defined, let $(D^{(M,U)})^{-1}_{w,w} = M^{-1}_{p(w),w}(D^{(M,U)})^{-1}_{p(w),p(w)} = 1$.

7. Calculate $\mathcal{B}_M = (D^{(M,U)})^{-1}\mathcal{B}_F$

8. OUTPUT $\mathcal{B}_M$

Algorithm 1 expands the set of matrices for which a sparsest basis of the null space can be found in optimal time. For an example, see Appendix A.

3 On the rank

In the previous section we proved that given a forest F and $M \in \mathcal{M}_{\mathbb{F},0}(F)$, $\mathcal{Null}(M)$ is a nonsingular diagonal multiplication of $\mathcal{Null}(A(F))$. In this section show that $\mathcal{Rank}(M)$ is a nonsingular diagonal multiplication of $\mathcal{Rank}(A(F))$. In order to do so, first we find a basis for the rank of M.

Let $v \notin \mathcal{Supp}(M)$, we define its *supported-neighborhood vector*, $\vec{s}_v$, as

$$\vec{s}_v(M) = \sum_{w \in \mathcal{Supp}(M) \cap N(v)} M_{v,w}\vec{e}_w,$$

where $\vec{e}_w$ denotes the vector with 1 in coordinate w and 0 elsewhere. For the matrix in

Figure 1 we have

$$\vec{s}_r(M) = 2\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\end{pmatrix} + (-2)\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}2\\0\\-2\\0\\0\\0\\0\end{pmatrix}$$

$$\vec{s}_t(M) = (-7)\begin{pmatrix}0\\0\\1\\0\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-7\\0\\1\\0\\0\end{pmatrix}$$

$$\vec{s}_v(M) = \vec{s}_w(M) = \begin{pmatrix}0\\0\\0\\0\\0\\0\\0\end{pmatrix}.$$

In [8] it was shown that

$$B(F) := \bigcup_{v \notin \mathcal{S}upp(F)} \{\vec{e}_v, \vec{s}_v(A(F))\} \setminus \{\vec{0}\}$$

is a basis for the rank of the adjacency matrix of F. We show the same result for $M \in \mathcal{M}_{\mathbb{F},0}(F)$.

Lemma 3.1. *If F is a forest and $M \in \mathcal{M}_{\mathbb{F},0}(F)$, then*

$$B(M) := \bigcup_{v \notin \mathcal{S}upp(M)} \{\vec{e}_v, \vec{s}_v(M)\} \setminus \{\vec{0}\}$$

is a basis for the rank of M.

Proof. It is easy to see that all columns of M can be written as linear combinations of

$$B(M) = \bigcup_{v \notin \mathcal{S}upp(M)} \{\vec{e}_v, \vec{s}_v(M)\} \setminus \{\vec{0}\}.$$

Hence $\mathcal{R}ank(M) \subset \mathcal{S}pan(B(M))$.

But $\dim(\mathcal{Null}(M)) = \dim(\mathcal{Null}(A(F)))$ by Theorem 2.3. Thus

$$\dim(\mathcal{Rank}(M)) = \dim(\mathcal{Rank}(A(F))) = |B(F)| = |B(M)|.$$

Therefore $B(M) = \bigcup_{v \notin \mathcal{Supp}(M)} \{\vec{e}_v, \vec{s}_v(M)\} \setminus \{\vec{0}\}$ is a basis for the rank of M. $\square$

Again, we work on a tree instead of a forest, because the rank is the direct sum of the ranks of the connected components.

Let T be a tree, $M \in \mathcal{M}_{\mathbb{F},0}(T)$ and v a vertex of T. For each vertex w let $\pi(v, w)$ be second vertex in vPw, where v is the first vertex of the path. We define the *v-normalization of M* as the nonsingular diagonal matrix with

$$C^{(M,v)}_{w_1,w_2} = \begin{cases} 1 & \text{if } w_1 = w_2 = v, \\ M_{v,\pi(v,w_1)} & \text{if } w_1 = w_2 \neq v, \\ 0 & \text{otherwise.} \end{cases}$$

For the matrix in Figure 1 we have

$$C^{(M,r)} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}$$

$$C^{(M,t)} = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$C^{(M,v)} = \begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 \end{pmatrix}$$

$$C^{(M,w)} = \begin{pmatrix} -5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We define the *rank-normalization of M, R^M*, as the product of $C^{(M,v)}$ over all vertices $v \notin \mathcal{Supp}(M)$.

$$R^M = \prod_{v \notin \mathcal{Supp}(M)} C^{(M,v)}.$$

For the matrix in Figure 1 we have

$$R^M = \begin{pmatrix} -315 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -105 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 105 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 35 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 175 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -75 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -45 \end{pmatrix}.$$

Lemma 3.2. *If T is a tree and $M \in \mathcal{M}_{\mathbb{F},0}(T)$, then $R^M\, \mathcal{Rank}(A(T)) = \mathcal{Rank}(M)$.*

Proof. Let $v \notin \mathcal{Supp}(M)$. We have

$$\begin{aligned} R^M \vec{s}_v(A(T)) &= \sum_{w \in \mathcal{Supp}(M) \cap N(v)} R^M \vec{e}_w \\ &= \sum_{w \in \mathcal{Supp}(M) \cap N(v)} C^{(M,v)} \prod_{\substack{u \notin \mathcal{Supp}(M), \\ u \neq v}} C^{(M,u)} \vec{e}_w. \end{aligned}$$

But if $w \in \mathcal{Supp}(M) \cap N(v)$ and $u \notin \mathcal{Supp}(M)$ with $u \neq v$, then $\pi(u,w) = \pi(u,v)$. Notice that $C^{(M,u)} e_w = M(u, \pi(u,v))\vec{e}_w$. Hence

$$\begin{aligned} R^M \vec{s}_v(A(T)) &= \sum_{w \in \mathcal{Supp}(M) \cap N(v)} C^{(M,v)} \prod_{\substack{u \notin \mathcal{Supp}(M), \\ u \neq v}} C^{(M,u)} \vec{e}_w \\ &= \sum_{w \in \mathcal{Supp}(M) \cap N(v)} C^{(M,v)} \prod_{\substack{u \notin \mathcal{Supp}(M), \\ u \neq v}} M(u, \pi(u,v)) \vec{e}_w \\ &= \prod_{\substack{u \notin \mathcal{Supp}(M), \\ u \neq v}} M(u, \pi(u,v)) \sum_{w \in \mathcal{Supp}(M) \cap N(v)} C^{(M,v)} \vec{e}_w. \end{aligned}$$

On the other hand, if $w \in N(v)$, $C^{(M,v)}\vec{e}_w = M(v,w)\vec{e}_w$. Therefore

$$\begin{aligned}
R^M \vec{s}_v(A(T)) &= \prod_{\substack{u \notin \mathcal{Supp}(M),\\ u \neq v}} M(u,\pi(u,v)) \sum_{w \in \mathcal{Supp}(M) \cap N(v)} C^{(M,v)}\vec{e}_w \\
&= \prod_{\substack{u \notin \mathcal{Supp}(M),\\ u \neq v}} M(u,\pi(u,v)) \sum_{w \in \mathcal{Supp}(M) \cap N(v)} M(v,w)\vec{e}_w \\
&= \prod_{\substack{u \notin \mathcal{Supp}(M),\\ u \neq v}} M(u,\pi(u,v))\vec{s}_v(M).
\end{aligned}$$

Hence $\mathcal{Span}(R^M \vec{s}_v(A(T))) = \mathcal{Span}(\vec{s}_v(M))$. It is easy to see that $\mathcal{Span}(R^M \vec{e}_v) = \mathcal{Span}(\vec{e}_v)$. Therefore $R^M\, \mathcal{Rank}(A(T)) = \mathcal{Rank}(M)$. □

If instead we consider a forest, then a diagonal matrix can be obtained by having $C^{(M,v)}_{w,w} = 1$ when v and w are in different connected components. Hence, we have the following.

Corollary 3.3. *Given a tree F and a matrix $M \in \mathcal{M}_{\mathbb{F},0}(F)$, $R^M\, \mathcal{Rank}(A(F)) = \mathcal{Rank}(M)$.*

The following result is a direct application, because R^M is nonsingular.

Corollary 3.4. *Let F be a forest, and $M, N \in \mathcal{M}_{\mathbb{F},0}(F)$. The vector $\vec{x}$ is in $\mathcal{Rank}(M)$ if and only if the vector $R^N(R^M)^{-1}\vec{x}$ is in $\mathcal{Rank}(N)$.*

4 Conclusion

There is a strong relation between the rank and the null space of a tree-patterned (acyclic) matrix with diagonal 0, and its underlying tree (forest). It would be interesting to study what happens when nonzero diagonal entries are allowed, or when a different graph is used. We conjecture that there will still be a strong relation, but it will not be as straightforward. For example, we conjecture that having nonzero diagonal entries only in the neighbors of $\mathcal{Supp}(F)$ should have no effect in the null space of the matrix.

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A Example of Algorithm 1

Consider the following matrix, with coefficients in $\mathbb{R}$,

$$M = \begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \\
5 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 4 & 0 & -6 & 0 & 8 & 0 & -16 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3 & 0 & 0 & 0 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 \\
11 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 13 & 7 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0
\end{pmatrix}.$$

Replacing every nonzero entry of M with a 1 we obtain the adjacency matrix

$$A(T) = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}$$

of T, the tree in Figure 2, where vertex v_i corresponds to row and column i. Thus, $M \in \mathcal{M}_{\mathbb{R}}(T)$.

We proceed to apply the algorithms from [7] to obtain $\mathcal{S}upp(T)$ and a sparsest basis for the null space of $A(T)$. We first obtain a maximum matching of T by matching a leaf to its neighbor, removing both vertices and repeating the process, until there are no edges remaining (see Figure 2). Next, we obtain $\mathcal{S}upp(T)$ by taking every vertex that is not incident to an edge in the maximum matching, and every vertex that can be reached from them by an even length path alternating between edges not in the matching and edges in the matching. In this way we get $\mathcal{S}upp(T) = \{v_1, v_3, v_5, v_7, v_9, v_{11}, v_{12}, v_{14}\}$ (see Figure 2).

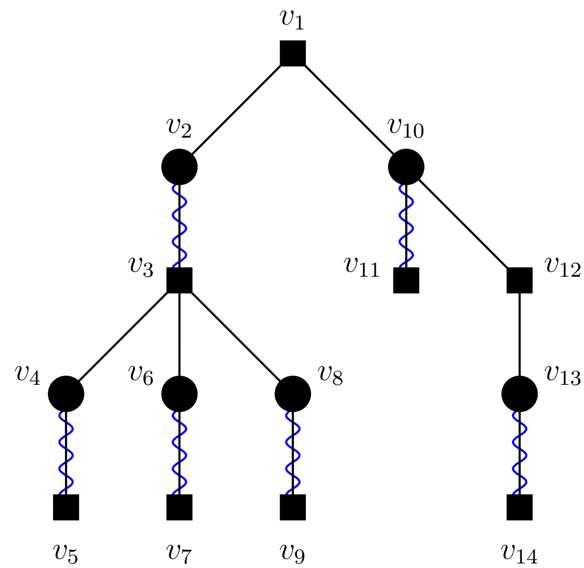

Figure 2: A tree T, with a maximum matching represented by decorated edges, and where vertices in $\mathcal{S}upp(T)$ are represented with squares, and vertices not in $\mathcal{S}upp(T)$ are represented with circles.

We now proceed to look for a sparsest basis for the null space of $A(T)$. We start by removing vertices that are neither in $\mathcal{S}upp(T)$ nor adjacent to a vertex in $\mathcal{S}upp(T)$. In the case of our example, there is no such vertex. Next, we need to root each connected component at some vertex in $\mathcal{S}upp(T)$. We do so with v_1. To obtain the basis we are going to obtain a different maximum matching. To do this we are going to give two weights to our remaining vertices, $\mathrm{wt_c}$ and $\mathrm{wt_p}$. We begin by giving each leaf ℓ, weight $\mathrm{wt_c}(\ell) = 1$. Let $C(v)$ denote the set of children of v (i.e., the set of vertices adjacent to v that are further from the root than v). A non-leaf vertex v is assigned $\mathrm{wt_c}$ once every vertex in $C(v)$ has been assigned a $\mathrm{wt_c}$. For a vertex v not in $\mathcal{S}upp(T)$, $\mathrm{wt_c}(v) = \min\{\mathrm{wt_c}(w) \mid w \in C(v)\}$. On the other hand, for a vertex v in $\mathcal{S}upp(T)$, $\mathrm{wt_c}(v) = 1 + \sum_{w\in C(v)} \mathrm{wt_c}(w)$. After this process the weights we have are

$$\begin{aligned}
\mathrm{wt_c}(v_5) =& \mathrm{wt_c}(v_7) = \mathrm{wt_c}(v_9) = \mathrm{wt_c}(v_{11}) = \mathrm{wt_c}(v_{14}) = 1\\
\mathrm{wt_c}(v_4) =& \mathrm{wt_c}(v_6) = \mathrm{wt_c}(v_8) = \mathrm{wt_c}(v_{13}) = 1\\
\mathrm{wt_c}(v_3) =& 4\\
\mathrm{wt_c}(v_{12}) =& 2\\
\mathrm{wt_c}(v_2) =& 4\\
\mathrm{wt_c}(v_{10}) =& 1\\
\mathrm{wt_c}(v_1) =& 6,
\end{aligned}$$

(see Figure 3). Once we have assigned a weight $\mathrm{wt_c}$ to the root, we start assigning $\mathrm{wt_p}$,

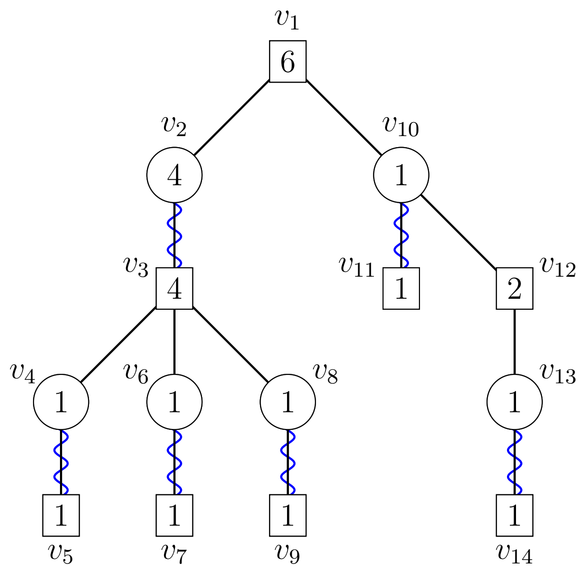

Figure 3: The tree T from Figure 2 with the assignment of $\mathrm{wt_c}$.

moving down through the tree again. If v is the root, we keep $\mathrm{wt_c}(v) = \mathrm{wt_p}(v)$. For a vertex w, let p_w be its parent. I.e., $p_w = v$ if and only if $w \in C(v)$. Once $\mathrm{wt_p}(p_w)$ has been assigned, we are ready to assign $\mathrm{wt_p}$ to w. For w not in $\mathcal{S}upp(T)$, if $\mathrm{wt_p}(p_w) - \mathrm{wt_c}(w) < \mathrm{wt_c}(w)$, then $\mathrm{wt_p}(w) = \mathrm{wt_p}(p_w) - \mathrm{wt_c}(w)$, and the edge in the matching containing w is replaced with the edge $\{w, p_w\}$ (this replacement is the reason why we assign the weights). Otherwise, $\mathrm{wt_p}(w) = \mathrm{wt_c}(w)$. For w in $\mathcal{S}upp(T)$, $\mathrm{wt_p}(w) = \mathrm{wt_c}(w) + \mathrm{wt_c}(p_w)$. Once the process is done for T, the values for $\mathrm{wt_p}$ are

$$\begin{aligned}
\mathrm{wt_c}(v_1) =& 6\\
\mathrm{wt_c}(v_2) =& 2\\
\mathrm{wt_c}(v_{10}) =& 1\\
\mathrm{wt_c}(v_3) =& 6\\
\mathrm{wt_c}(v_{12}) =& 3\\
\mathrm{wt_c}(v_4) =& \mathrm{wt_c}(v_6) = \mathrm{wt_c}(v_8) = \mathrm{wt_c}(v_{13}) = 1\\
\mathrm{wt_c}(v_5) =& \mathrm{wt_c}(v_7) = \mathrm{wt_c}(v_9) = \mathrm{wt_c}(v_{11}) = \mathrm{wt_c}(v_{14}) = 2,
\end{aligned}$$

and the edge $\{v_2, v_3\}$ in the matching is changed for the edge $\{v_1, v_2\}$ (see Figure 4). To obtain the basis, we form vectors as follows. Take a vertex v that is in none of the edges of the matching. Start by placing a 1 in the coordinate of vertex v, i.e., $\vec{x}_v = 1$. If the edge $\{u, w\}$ is in the matching, $\vec{x}_u$ has not been assigned yet, and $\vec{x}_v \neq 0$ for some vertex v adjacent to w, then let $\vec{x}_u = -\vec{x}_v$. Once this process stops, let $\vec{x}_w = 0$ for every vertex w whose coordinate was not assigned. In this way, we form a vector $\vec{x}$ starting with v_3, by

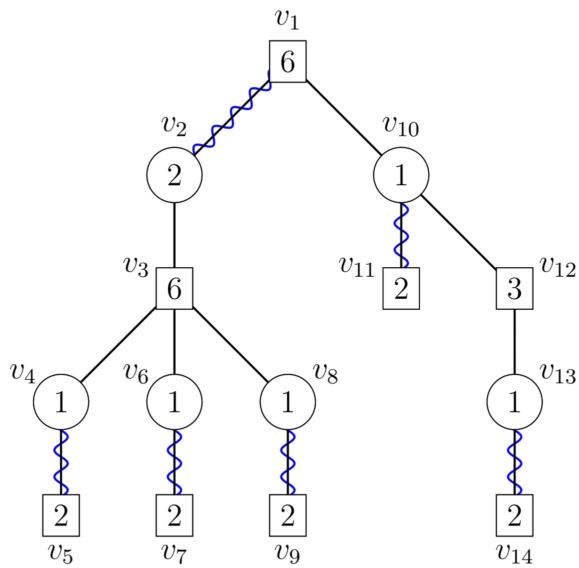

Figure 4: The tree T from Figure 2 with the assignment of $\mathrm{wt_p}$, where the maximum matching was changed.

setting

$$\begin{aligned}\vec{x}_{v_3} &= 1\\ \vec{x}_{v_1} &= \vec{x}_{v_5} = \vec{x}_{v_7} = \vec{x}_{v_9} = -1\\ \vec{x}_{v_{11}} &= 1,\end{aligned}$$

and $\vec{x}_w = 0$ for all remaining coordinates. Similarly, starting with v_{12}, we form vector $\vec{y}$ with

$$\begin{aligned}\vec{y}_{v_{12}} &= 1\\ \vec{y}_{v_{11}} &= \vec{y}_{v_{14}} = -1\end{aligned}$$

and $\vec{y}_w = 0$ for all remaining coordinates. Thus,

$$\mathcal{B}_T = \{(-1,0,1,0,-1,0,-1,0,-1,0,1,0,0,0)^T, (0,0,0,0,0,0,0,0,0,0,-1,1,0,-1)^T\}$$

is a sparsest basis for the null space of $A(T)$.

Now, for each connected component with at least one vertex in $\mathcal{Supp}(A(T))$, we need to choose a vertex in $\mathcal{Supp}(A(T))$. As we have just one connected component, we let v_1 be the vertex. Applying the properties given in Lemma 2.2, we obtain

$$\begin{array}{lll}
D^{(M,v_1)}_{v_1,v_1} = 1 & & \\
D^{(M,v_1)}_{v_2,v_2} = \frac{1}{5} & D^{(M,v_1)}_{v_{10},v_{10}} = \frac{1}{11} & \\
D^{(M,v_1)}_{v_3,v_3} = -\frac{1}{5} & D^{(M,v_1)}_{v_{12},v_{12}} = \frac{7}{11} & D^{(M,v_1)}_{v_{11},v_{11}} = \frac{13}{11} \\
D^{(M,v_1)}_{v_4,v_4} = -\frac{1}{10} & D^{(M,v_1)}_{v_{13},v_{13}} = \frac{7}{33} & \\
D^{(M,v_1)}_{v_5,v_5} = -\frac{2}{5} & D^{(M,v_1)}_{v_{14},v_{14}} = \frac{35}{33} & \\
D^{(M,v_1)}_{v_6,v_6} = \frac{1}{15} & D^{(M,v_1)}_{v_8,v_8} = -\frac{1}{25} & \\
D^{(M,v_1)}_{v_7,v_7} = -\frac{4}{15} & D^{(M,v_1)}_{v_9,v_9} = -\frac{6}{25} &
\end{array}$$

and

$$\left(D^{(M,v_1)}\right)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{5}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{15}{4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -25 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{25}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 11 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{13} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{11}{7} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{33}{7} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{33}{35}
\end{pmatrix},$$

and

$$\begin{aligned}
\mathcal{B}_M &= (D^{(M,v_1)})^{-1}\mathcal{B}_T \\
&= \left\{ \begin{array}{l} \left(-1, 0, -5, 0, \frac{5}{2}, 0, \frac{15}{4}, 0, \frac{25}{6}, 0, \frac{11}{13}, 0, 0, 0\right)^T, \\ \left(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, -\frac{11}{13}, \frac{11}{7}, 0, -\frac{33}{35}\right)^T \end{array} \right\}
\end{aligned}$$

is a sparsest basis for $\mathcal{N}ull(M)$.

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