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On the distance spectrum of certain distance biregular graphs

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Abstract

In this article we present an infinite family of bipartite distance biregular graphs having an arbitrarily large diameter and whose distance matrices have exactly four distinct eigenvalues. This result answers a question posed by F. Atik and P. Panigrahi in *On the distance spectrum of distance regular graphs* (Linear Algebra and its Applications, 478 (2015), pp. $256-273$) about the existence of connected graphs with diameter d that are not distance regular, whose distance matrix has less than $d+1$ distinct eigenvalues.

1 Introduction

If $G=(V,E)$ is a simple connected graph where V is the set of vertices and E the set of edges, it is well known that the set of distinct eigenvalues of the adjacency matrix A has at least $d+1$ elements where d is the diameter of G. The same holds for the Laplacian and the signless Laplacian matrices. It is natural to ask if this result is true for the distance matrix D of the graph, that is, if the distance matrix of a connected simple graph of diameter d has at least $d+1$ distinct eigenvalues. In [7], the authors raise this question. Atik and Panigrahi in [3] show that distance regular graphs with diameter d have at most $d+1$ distinct eigenvalues. They also exhibit a class of distance regular graphs (Johnson graphs) having an arbitrarily large diameter, but the distance matrix has only three distinct eigenvalues. The authors ask if there are other connected graphs besides the distance regular graphs with diameter d, whose distance matrix has less than $d+1$ distinct eigenvalues. In the present work, we answer positively to this question, presenting an infinite family of distance biregular graphs having an arbitrarily large diameter such that their distance matrices have exactly four distinct eigenvalues. As in the case of the regular graphs presented in [3], the number

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of distinct distance eigenvalues for the graphs in this family can be much smaller than the diameter plus one.

We note that in [2] is presented a family of connected, biregular bipartite graphs, for the particular case when the diameter is four, such that their distance matrices have exactly four distinct eigenvalues.

Our main result is Theorem 3.1, in which the spectrum of particular distance biregular graphs, having an arbitrarily large diameter, is obtained. The result of [2] can be seen as a particular case of this theorem.

We emphasize that although Theorem 3.1 is a generalization of the result presented in [2], the proof follows an entirely different path of the particular case, since the techniques used there cannot be used in the more general setting. In our case, the employed tools combine combinatorial and linear algebra techniques.

This paper is structured as follows: after this introduction, in the second section, we present concepts and results on distance regular and distance biregular graphs. The third section is devoted to Theorem 3.1, presenting lemmas and propositions necessary for the proof of the central result. In addition, to facilitate the understanding of the reasoning followed in the proofs, some statements have been proved separately, in Appendix A.

2 Distance regular and distance biregular graphs

In this section we define distance regular and distance biregular graphs, present some of their properties and give several examples of such graphs. The definitions and results presented here serve as a basis for the results of the next section.

2.1 Distance regular graphs

Let $G = (V, E)$ be a simple connected graph with diameter d. For a fixed vertex $x \in V$, and for all $i \in \{0, 1, \ldots, d\}$, we define the set $G_i(x)$ as follows

$$G_i(x) = \{y \in V : d(x, y) = i\}.$$

Definition 2.1. A simple connected graph G with diameter d, is *distance regular* if it is a regular graph of degree r and if there exist integer numbers $c_1, \ldots, c_d, b_0, b_1, \ldots, b_{d-1}$ such that if $x, y \in V$ with $d(x, y) = i$, the following hold

1. $c_i = |G_{i-1}(x) \cap G_1(y)|$, for all $i \in \{1, \ldots, d\}$;
2. $b_i = |G_{i+1}(x) \cap G_1(y)|$, for all $i \in \{0, 1, \ldots, d-1\}$.

Definition 2.2. The sequence $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$, where $b_0 = r$ and $c_1 = 1$, is called the *intersection array* of the distance regular graph G and the numbers c_i, b_i and $a_i := r - b_i - c_i$ are the *intersection numbers.*

For a distance regular graph G it is well known [4] that its adjacency matrix has exactly $d+1$ distinct eigenvalues. In fact, the eigenvalues of G are precisely the eigenvalues of the $(d+1)\times(d+1)$ matrix T given by the intersection numbers as follows

$$T = \begin{pmatrix} 0 & b_0 & 0 & 0 & 0 & 0 \\ c_1 & a_1 & b_1 & 0 & 0 & 0 \\ 0 & c_2 & \cdot & \cdot & & 0 \\ 0 & 0 & \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \cdot & \cdot & b_{d-1} \\ 0 & 0 & 0 & 0 & c_d & a_d \end{pmatrix}.$$

We denote by D-spectrum, the spectrum of the distance matrix D.

For all $i,j,h \in \{0,1,\ldots,d\}$ and $x,y \in V$ with $d(x,y)=i$, we define:

$$p^i_{jh} := |G_j(x) \cap G_h(y)|.$$

The numbers p^i_{jh} can be interpreted as the number of vertices z such that $d(z,x)=j$ and $d(z,y)=h$.

The next result implies that, for a distance regular graph, the number of distinct eigenvalues of the distance matrix D is at most $d+1$.

Theorem 2.3. *[3, Theorem 3.1] Let G be a distance regular graph with diameter d and let D be the distance matrix of G. If $B=(b_{ij}) \in M_{(d+1)\times(d+1)}$ is the matrix given by $b_{ij}=\sum_{m=0}^{d} m p^i_{mj}$, then D and B have the same set of eigenvalues.*

2.2 Distance biregular graphs

Distance biregular graphs can be seen as the bipartite analogue of distance regular ones, in the following sense.

Definition 2.4. A bipartite connected graph $G=(V_1 \cup V_2, E)$, with diameter d, is called *distance biregular* if vertices belonging to the same component have the same intersection array.

The intersection numbers of a vertex $x \in V = V_1 \cup V_2$ is defined as

1. $c_i(x) = |G_{i-1}(x) \cap G_1(y)|$, for all $i \in \{1,\ldots,d\}$;
2. $b_i(x) = |G_{i+1}(x) \cap G_1(y)|$, for all $i \in \{1,\ldots,d-1\}$,

for every $y \in V$ such that $d(x,y)=i$.

Note that we have to require that the numbers above do not depend on y but only on the distance i.

A first example of a distance biregular graphs is the following (more examples can be found in [6]).

Example 2.5. The complete bipartite graph $K_{a,b}$ is a distance biregular graph whose intersection array is given by

$$i(V_1) = \{a; 1, a\};$$
$$i(V_2) = \{b; 1, b\}.$$

We now introduce the graphs $G_{(k,p)}$ which will be studied in the next section.

Through this work we use the following notations: for a positive integer k, $[k] := \{1, 2, \ldots, k\}$. Given a set A and a positive integer m, $[A, m]$ denotes the set of all subsets of A of order m.

Definition 2.6. Let k, p be positive integers such that $2p+1 \leq k$ and denote by $V_p := [[k], p]$ and $V_{p+1} := [[k], p+1]$ the sets of all subsets of $[k]$ with p and $p+1$ elements, respectively. We define the graph $G_{(k,p)} := G(V, E)$ where:

1. The set of vertices is $V = V_p \cup V_{p+1}$;
2. The set of edges is $E = \{\{x, y\} \subset V; x \subset y \text{ or } y \subset x\}$.

From the definition we have that

$$|V| = \binom{k}{p} + \binom{k}{p+1} = \binom{k+1}{p+1} \text{ and } d = \operatorname{diam}(G_{(k,p)}) = \begin{cases} 2p+1, & \text{if } 2p+1 = k; \\ 2p+2, & \text{if } 2p+1 < k. \end{cases}$$

Moreover, for all $x \in V$ the degree of x is

$$\deg(x) = \begin{cases} k-p, & \text{if } x \in V_p; \\ p+1, & \text{if } x \in V_{p+1}. \end{cases}$$

Remark 2.7. Notice that, when $2p+1 = k$, the graph $G_{(2p+1,p)}$ is distance regular. In fact, in this case the intersection array is given by

$$i(V_p) = \{p+1; 1, 1, 2, 2, 3, \ldots, p, p, p+1\} = i(V_{p+1}).$$

The graph $G_{(2p+1,p)}$ is called double odd and its D-spectrum is characterized in [1].

Remark 2.8. Note that if $2p+1 > k$ we can still define a graph $G_{(k,p)}$ as in Definition 2.6. However, in this case we have that $G_{(k,p)}$ is isomorphic to $G_{(k,k-p-1)}$ and $2(k-p-1)+1 < k$.

By the results in [6] we have that, for all $k, p \in \mathbb{N}$ such that $2p+1 < k$, the graph $G_{(k,p)}$ is distance biregular. Moreover, its intersection array is

$$i(V_p) = \{k-p; 1, 1, 2, 2, 3, \ldots, p, p, p+1\};$$
$$i(V_{p+1}) = \{p+1; 1, 1, 2, 2, 3, \ldots, p, p, p+1, p+1\}.$$

The adjacency matrix of a distance biregular graph with diameter d has also $d+1$ distinct eigenvalues as shown in [6]. Here we prove that the distance matrices D of the graphs $G_{(k,p)}$ have only 4 distinct eigenvalues regardless of its diameter.

3 The main theorem

In this section we completely characterize the D-spectrum of the graph $G_{(k,p)}$ defined in the previous section. We denote the spectrum of D by $\text{spect}_D(G)$, a matrix whose entries of the first row are the eigenvalues of D while the entries of the second row are their respective multiplicities.

Theorem 3.1. *Let k, p be positive integers such that $2p + 1 \leq k$. Then the D-spectrum of the graph $G_{(k,p)}$ is given by*

$$\text{spect}_D(G) = \begin{bmatrix} -2\binom{k-1}{p} & \lambda_1 & 0 & \lambda_2 \\ \\ k-1 & 1 & \binom{k+1}{p+1} - k - 1 & 1 \end{bmatrix},$$

where λ_1 and λ_2 are the roots of the polynomial $x^2 - 2(k-1)\binom{k-1}{p}x - \binom{k}{p}\binom{k}{p+1}$.

Prior proving Theorem 3.1, we next recall a necessary result from matrix theory.

Lemma 3.2. *[5, Lemma 2.3.1] Let B be a quotient matrix of a square matrix D corresponding to an equitable partition. Then the spectrum of D contains the spectrum of B.*

The idea to prove Theorem 3.1 is to construct an equitable partition P on the set of vertices of $G_{(k,p)}$ and prove that, up to multiplicities, the set of eigenvalues of the quotient matrix coincides with the set of the D-eigenvalues.

From now on, let $G = G_{(k,p)}$ where $k, p \in \mathbb{N}$ and $2p + 1 \leq k$. We fix a vertex $x = \{\ell_1, \ldots, \ell_{p+1}\} \in V_{p+1}$. The set of vertices V of $G_{(k,p)}$ can be partitioned as follows

$$P = \{G_i(x) \subset V : i = 0, 1, \ldots, d\}, \tag{3.1}$$

where $G_i(x)$ is the subset of V that consists of the vertices whose distance to x is equal to i.

We begin with some necessary lemmas and propositions.

Lemma 3.3. *Let $y, z \in V$. Then the following properties hold.*

1. *If $y, z \in V_p$, then $d(y, z) = 2(p - |y \cap z|)$.*
2. *If $y \in V_p$ and $z \in V_{p+1}$, then $d(y, z) = 2(p - |y \cap z|) + 1$.*
3. *If $y, z \in V_{p+1}$, then $d(y, z) = 2(p + 1 - |y \cap z|)$.*

Proof. We begin by proving item 1. A path $y = x_0, x_1, \ldots, x_{2m} = z$ in G is equivalent to a sequence of inclusions

$$y = x_0 \subset x_1 \supset x_2 \subset x_3 \supset x_4 \subset \ldots \subset x_{2m-1} \supset x_{2m} = z,$$

where $|x_i| = p$ if i is even and $|x_i| = p + 1$ if i is odd. In this way, at each step $x_{2i} \subset x_{2i+1}$ we have to add an element to x_{2i} while at each step $x_{2i+1} \supset x_{2i+2}$ we have to remove an element from x_{2i+1}. The shortest way to do that is to add the $|z \setminus y| = p - |y \cap z|$ elements of z that are not in y and remove $|y \setminus z| = p - |y \cap z|$ elements of y that are not in z. So, $d(y, z) = 2(p - |y \cap z|)$.

The other cases are treated in a similar way.

□

Lemma 3.4. *Let W be a set of order n and consider $U \subseteq W$ such that $|U| = m$, where $m \in \mathbb{N} \cup \{0\}$. For all integers $0 \le \overline{r} \le n$, it holds*

$$\sum_{A \in [W,\overline{r}]} |A \cap U| = \sum_{\overline{s}=0}^{m} \overline{s}\binom{m}{\overline{s}}\binom{n-m}{\overline{r}-\overline{s}} = m\binom{n-1}{\overline{r}-1}.$$

Proof. Since

$$\overline{s}\binom{m}{\overline{s}}\binom{n-m}{\overline{r}-\overline{s}} = m\binom{m-1}{\overline{s}-1}\binom{n-m}{\overline{r}-\overline{s}},$$

we have

$$\sum_{\overline{s}=0}^{m} \overline{s}\binom{m}{\overline{s}}\binom{n-m}{\overline{r}-\overline{s}} = m\sum_{\overline{s}=0}^{m}\binom{m-1}{\overline{s}-1}\binom{n-m}{\overline{r}-\overline{s}} = m\binom{n-1}{\overline{r}-1},$$

where the last equality follows from the Vandermonde's identity (see Appendix A). □

Proposition 3.5. *The partition P defined in* (3.1) *gives an equitable partition of D, the distance matrix of the graph $G = G_{(k,p)}$.*

Proof. Let $B_{i,j}$ be the submatrices of D, of order $|G_i(x)| \times |G_j(x)|$ with $i, j \in \{0, 1, \ldots, d\}$, obtained from the partition P. Then

$$D = \begin{pmatrix} B_{0,0} & B_{0,1} & \ldots & B_{0,d} \\ B_{1,0} & B_{1,1} & \ldots & B_{1,d} \\ \vdots & \vdots & \ddots & \vdots \\ B_{d,0} & B_{d,1} & \ldots & B_{d,d} \end{pmatrix}.$$

Since x is in V_{p+1}, we have the following possibilities for $d(x, y)$ when y varies in V.

1. $d(x, y) = 2r - 1$ with $1 \le r \le p + 1$. In this case $y \in V_p$ and y is of the form $y = Y_1 \cup Y_2$ where $Y_1 \in [x, p + 1 - r]$ and $Y_2 \in [[k] \setminus x, r - 1]$. Notice that x and y have $p+1-r$ elements in common, and these elements will form the set Y_1. The other $r-1$ elements of y must be chosen in the set $[k]\setminus x = \{1, \ldots, k\}\setminus x$ and they will form Y_2.

2. $d(x, y) = 2r$ with $1 \le r \le \lfloor \frac{d}{2} \rfloor$. In this case $y \in V_{p+1}$ and y is of the form $y = Y_1 \cup Y_2$, where $Y_1 \in [x, p+1-r]$ and $Y_2 \in [[k]\setminus x, r]$ are defined analogously as in the item above.

Let $y \in G_i(x)$. For each j we define

$$b_{ij}(y) := \sum_{z \in G_j(x)} d(y, z).$$

That is, $b_{ij}(y)$ is the summation of the coefficients of the row y of the sub-matrix $B_{i,j}$.

To prove that the partition is equitable, we need to prove that, for all $i, j \in \{0, \ldots, d\}$, the number $b_{ij} := b_{ij}(y)$ does not depend on $y \in G_i(x)$. There are 4 cases to consider:

1. The numbers i, j are odd, so we write $i = 2r-1$ and $j = 2s-1$ where $r, s \in \{1, \ldots, p+1\}$;
2. The number i is even, while j is odd, so we write $i = 2r$ and $j = 2s-1$ where $r \in \{0, \ldots, \lfloor \frac{d}{2} \rfloor\}$ and $s \in \{1, \ldots, p+1\}$;
3. The number i is odd, while j is even, so we write $i = 2r-1$ and $j = 2s$ where $r \in \{1, \ldots, p+1\}$ and $s \in \{0, \ldots, \lfloor \frac{d}{2} \rfloor\}$;
4. The numbers i, j are even, so we write $i = 2r$ and $j = 2s$ where $r, s \in \{0, \ldots, \lfloor \frac{d}{2} \rfloor\}$.

We start with the first case. Then, $y = Y_1 \cup Y_2$ and $z = Z_1 \cup Z_2$ for some

$$Y_1 \in [x, p+1-r], \quad Y_2 \in [[k] \setminus x, r-1], \quad Z_1 \in [x, p+1-s] \quad \text{and} \quad Z_2 \in [[k] \setminus x, s-1].$$

From Lemma 3.3, we obtain

$$b_{ij}(y) = \sum_{z \in G_j(x)} d(y, z) = \sum_{Z_1} \sum_{Z_2} d(y, Z_1 \cup Z_2) = \sum_{Z_1} \sum_{Z_2} 2\big(p - |(Z_1 \cup Z_2) \cap y|\big),$$

where the sum runs through all $Z_1 \in [x, p+1-s]$ and $Z_2 \in [[k] \setminus x, s-1]$.

Since $Z_1 \cap Y_2 = \emptyset = Z_2 \cap Y_1$, we have that $|(Z_1 \cup Z_2) \cap y| = |Z_1 \cap Y_1| + |Z_2 \cap Y_2|$. Therefore,

$$\begin{aligned}
b_{ij}(y) &= \sum_{Z_1} \sum_{Z_2} 2\big(p - |Z_1 \cap Y_1| - |Z_2 \cap Y_2|\big) \\
&= 2\Bigg(p\Big|[x, p+1-s]\Big| \cdot \Big|[[k] \setminus x, s-1]\Big| \cdot \Big|[[k] \setminus x, s-1]\Big| \sum_{Z_1} \Big|Z_1 \cap Y_1\Big| - \\
&\quad - \Big|[x, p+1-s]\Big| \sum_{Z_2} \Big|Z_2 \cap Y_2\Big|\Bigg) \\
&= 2\Bigg(p\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - \binom{k-p-1}{s-1} \sum_{Z_1} |Z_1 \cap Y_1| - \\
&\quad - \binom{p+1}{p+1-s} \sum_{Z_2} |Z_2 \cap Y_2|\Bigg).
\end{aligned}$$

To compute the summation on Z_1 and Z_2, we use Lemma 3.4. We have

$$\begin{aligned}
b_{ij}(y) &= 2\Bigg(p\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - (p+1-r)\binom{p}{p-s}\binom{k-p-1}{s-1} - \\
&\quad - (r-1)\binom{p+1}{p+1-s}\binom{k-p-2}{s-2}\Bigg) \\
&= 2p\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - 2(p+1-r)\binom{p}{p-s}\binom{k-p-1}{s-1} - \\
&\quad - 2(r-1)\binom{p+1}{p+1-s}\binom{k-p-2}{s-2}.
\end{aligned}$$

Therefore, we conclude that $b_{ij} = b_{ij}(y)$ does not depend on the choice of $y \in G_i(x)$. The other cases are treated in a similar fashion. □

As an immediate consequence of Proposition 3.5, we have the following result.

Proposition 3.6. *For all $i, j \in \{0, \ldots, d\}$, the value b_{ij} is given by*

i	j	b_{ij}
$2r-1$	$2s-1$	$2p\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - 2(p+1-r)\binom{k-p-1}{s-1}\binom{p}{p-s}$ $-2(r-1)\binom{p+1}{p+1-s}\binom{k-p-2}{s-2}$
$2r$	$2s-1$	$(2p+1)\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - 2(p+1-r)\binom{k-p-1}{s-1}\binom{p}{p-s}$ $-2r\binom{p+1}{p+1-s}\binom{k-p-2}{s-2}$
$2r-1$	$2s$	$(2p+1)\binom{p+1}{p+1-s}\binom{k-p-1}{s} - 2(p+1-r)\binom{k-p-1}{s}\binom{p}{p-s}$ $-2(r-1)\binom{p+1}{p+1-s}\binom{k-p-2}{s-1}$
$2r$	$2s$	$2(p+1)\binom{p+1}{p+1-s}\binom{k-p-1}{s} - 2(p+1-r)\binom{k-p-1}{s}\binom{p}{p-s}$ $-2r\binom{p+1}{p+1-s}\binom{k-p-2}{s-1}$

Let D be the distance matrix of $G = G_{(k,p)}$, where we order the vertices of G by distance to the vertex x, that is, we choose an ordering of the vertices where each vertex in $G_i(x)$ comes before any vertex of $G_{i+1}(x)$, for every $i = 0, 1, \ldots, d$. If $B = (b_{ij})$ is the quotient matrix of D, obtained from the equitable partition P, then we have the following characterization of the eigenvalues of B.

Proposition 3.7. *Let k and p be positive integers such that $2p+1 \leq k$. Then the eigenvalues of the quotient matrix B are*

$$\left\{\lambda_1, 0, \lambda_2, -2\binom{k-1}{p}\right\},$$

where λ_1 and λ_2 are the roots of the polynomial

$$x^2 - 2(k-1)\binom{k-1}{p}x - \binom{k}{p}\binom{k}{p+1}.$$

Proof. By Proposition 3.6 we have the following values for $b_{i+1,j} - b_{ij}$ for all $i, j \in \{0, \ldots, d\}$.

i	j	$b_{i+1,j} - b_{ij}$
$2r-1$	$2s-1$	$\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - 2\binom{p+1}{p+1-s}\binom{k-p-2}{s-2}$
$2r-1$	$2s$	$\binom{p+1}{p+1-s}\binom{k-p-1}{s} - 2\binom{p+1}{p+1-s}\binom{k-p-2}{s-1}$
$2r$	$2s-1$	$-\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} + 2\binom{k-p-1}{s-1}\binom{p}{p-s}$
$2r$	$2s$	$-\binom{p+1}{p+1-s}\binom{k-p-1}{s} + 2\binom{k-p-1}{s}\binom{p}{p-s}$

Assume that $L_i = (b_{ij})_{j\in\{0,\ldots,d\}}$ is the i-th row of B. Then

- For all $r \in \{0, \ldots, p+1\}$, $L_{2r+1} - L_{2r} = F_0$, where F_0 is the row vector defined by

$$F_{0,j} := \begin{cases} -\binom{p+1}{p+1-s}\binom{k-p-1}{s} + 2\binom{k-p-1}{s}\binom{p}{p-s}, & \text{if } j = 2s; \\ -\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} + 2\binom{k-p-1}{s-1}\binom{p}{p-s}, & \text{if } j = 2s-1. \end{cases}$$

- For all $r \in \{0, \ldots, \lfloor \frac{d}{2} \rfloor\}$, $L_{2r} - L_{2r-1} = F_1$, where F_1 is the row vector defined by

$$F_{1,j} := \begin{cases} \binom{p+1}{p+1-s}\binom{k-p-1}{s} - 2\binom{p+1}{p+1-s}\binom{k-p-2}{s-1}, & \text{if } j = 2s; \\ \binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - 2\binom{p+1}{p+1-s}\binom{k-p-2}{s-2}, & \text{if } j = 2s-1. \end{cases}$$

We also have that $L_0 = (L_{0,j})_{j\in\{0,\ldots,d\}}$, where

$$L_{0,j} := b_{0j} = \begin{cases} 2(p+1)\binom{p+1}{p+1-s}\binom{k-p-1}{s} - 2(p+1)\binom{k-p-1}{s}\binom{p}{p-s}, & \text{if } j = 2s; \\ (2p+1)\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - 2(p+1)\binom{k-p-1}{s-1}\binom{p}{p-s}, & \text{if } j = 2s-1. \end{cases}$$

As the row vectors F_0 and F_1 do not depend on r, the matrix B can be written as

$$B = \begin{pmatrix} L_0 \\ L_0 + F_0 \\ L_0 + F_0 + F_1 \\ L_0 + 2F_0 + F_1 \\ L_0 + 2F_0 + 2F_1 \\ \vdots \\ L_0 + (p+1)F_0 + pF_1 \\ L_0 + (p+1)F_0 + (p+1)F_1 \end{pmatrix}, \quad \text{if } d = 2p+2.$$

If $d = 2p+1$, which means $k = 2p+1$, we exclude the last row.

In order to compute the eigenvalues of B, we compute the eigenvalues of its transpose matrix B^t whose columns will be denoted by C_i with $i = 0, \ldots, d+1$.

Observe that $v = (v_0, v_1, \ldots, v_d) \in \mathbb{R}^{d+1}$. Then

$$\begin{aligned} B^t v &= v_0 \cdot C_0 + v_1 \cdot C_1 + \ldots + v_{2p+2} \cdot C_{2p+2} \\ &= v_0 \cdot L_0 + v_1 \cdot (L_0 + F_0) + v_2 \cdot (L_0 + F_0 + F_1) + \ldots + \\ &\quad + v_{2p+2} \cdot (L_0 + (p+1)F_0 + (p+1)F_1) \\ &= (v_0 + v_1 + \ldots + v_{2p+2}) \cdot L_0 + \\ &\quad + (v_1 + v_2 + 2v_3 + 2v_4 + \ldots + (p+1)v_{2p+1} + (p+1)v_{2p+2}) \cdot F_0 + \\ &\quad + (v_2 + v_3 + 2v_4 + 2v_5 + \ldots + pv_{2p+1} + (p+1)v_{2p+2}) \cdot F_1, \end{aligned}$$

for the case $d = 2p+2$, or

$$\begin{aligned} B^t v &= v_0 \cdot C_0 + v_1 \cdot C_1 + \ldots + v_{2p+1} \cdot C_{2p+1} \\ &= (v_0 + v_1 + \ldots + v_{2p+1}) \cdot L_0 + \\ &\quad + (v_1 + v_2 + 2v_3 + 2v_4 + \ldots + (p+1)v_{2p+1}) \cdot F_0 + \\ &\quad + (v_2 + v_3 + 2v_4 + 2v_5 + \ldots + pv_{2p+1}) \cdot F_1, \end{aligned}$$

when $d = 2p+1$.

In both cases the image of B^t, denoted by $\mathrm{Im}(B^t)$ (considering B^t as a linear transformation), is generated by the vectors L_0, F_0 and F_1. Then 0 is an eigenvalue of B^t with multiplicity at least $d-2$. We restrict B^t to the invariant space $\mathrm{Span}(L_0, F_0, F_1)$ generated by L_0, F_0 and F_1 and denote by $\overline{B} := B^t|_{\mathrm{Span}(L_0,F_0,F_1)}$.

Claim 3.8. With the above notations, the following equality holds.

$$\overline{B} = \binom{k-1}{p} \begin{pmatrix} \frac{k(2k-2p-1)}{k-p} & -\frac{k(k-2p-1)}{(k-p)(p+1)} & \frac{k(k-2p-1)}{(k-p)(p+1)} \\ (p+1)(2k-2p-1) & 2p-k & k-2p-2 \\ \frac{(k-p-1)(2kp-2p^2+2k-3p)}{k-p} & \frac{(k-p-1)(2p-k)}{k-p} & \frac{k^2-3kp+2p^2-3k+4p}{k-p} \end{pmatrix}.$$

The proof of this claim can be found in the Appendix A.

Let $M = \overline{B}/\binom{k-1}{p}$ be the matrix obtaining from $\overline{B}$ by dividing each coefficient by $\binom{k-1}{p}$. The characteristic polynomial of M is

$$(x+2)\Big(x^2 - (2k-2)x - \frac{k^2}{(k-p)(p+1)}\Big),$$

and from this, it follows that the eigenvalues of B are the ones claimed in the proposition. Indeed, since 0 is an eigenvalue with multiplicity at least $d-2$ and we have other 3 eigenvalues coming from $\overline{B}$, these are all eigenvalues of B. □

In the following Propositions we give lower bounds for the dimensions of the Kernel of D, denoted by $\ker(D)$, and $\ker(D + 2\binom{k-1}{p}I)$, that is, lower bounds for the multiplicities of the eigenvalues 0 and $-2\binom{k-1}{p}$ of D, respectively.

Proposition 3.9. *Let D be the distance matrix of the graph $G = G_{(k,p)}$. Consider the vectors given as follows*

$$v_\ell := (v_{\ell,z})_{z\in V}, \qquad \textit{where} \qquad v_{\ell,z} := \begin{cases} 0, & \textit{if } \ell \in z; \\ 1, & \textit{if } \ell \notin z, \end{cases}$$

for all $\ell \in \{1,\dots,k\}$ and

$$v_0 = (v_{0,z})_{z\in V}, \qquad \textit{where} \qquad v_{0,z} = \begin{cases} 0, & \textit{if } z \in V_p; \\ 1, & \textit{if } z \in V_{p+1}. \end{cases}$$

Consequently, $\mathrm{Im}(D) \subseteq \mathrm{Span}(v_0, v_1, \dots, v_k)$. *In particular* $\dim(\ker(D)) \geq \binom{k+1}{p+1} - k - 1$.

Proof. Since the columns of the matrix D generate $\mathrm{Im}(D)$, it is sufficient to see that all columns of D belong to $\mathrm{Span}(v_0, v_1, \dots, v_k)$. Note that if $w := \sum_{\ell=1}^{k} v_\ell$, then w has entries

$$w_z := \sum_{\ell=1}^{k} v_{\ell,z} = k - |z|,$$

that is, w is of the form

$$w = (w_z)_{z\in V} \quad \text{where} \quad w_z = \begin{cases} k-p, & \text{if } z \in V_p; \\ k-(p+1), & \text{if } z \in V_{p+1}. \end{cases}$$

The vector v'_0, defined by

$$v'_{0,z} := \begin{cases} 1, & \text{if } z \in V_p; \\ 0, & \text{if } z \in V_{p+1}, \end{cases}$$

is a linear combination of the vectors $v_0, \ldots, v_k$. In fact, we have that $w = (k-p)v'_0 + (k-p-1)v_0$, that is,

$$v'_0 = \frac{w}{(k-p)} - \frac{(k-p-1)}{(k-p)}v_0 = \frac{1}{(k-p)}\sum_{i=1}^{\ell} v_\ell - \frac{(k-p-1)}{(k-p)}v_0.$$

For all $y, z \in V$, we define C_y as the column of D indexed by the vertex y and $c_{y,z}$, the entry of C_y corresponding to the row of D indexed by z. For y fixed, $c_{y,z} = d(y,z)$ holds for every $z \in V$. Since $V = V_p \cup V_{p+1}$, we need to analyze four cases:

1. Let $y = \{\ell_1, \ldots, \ell_p\} \in V_p$ with $\ell_1, \ldots, \ell_p \in \{1, \ldots, k\}$. Then:

   (a) For $z \in V_p$, by Lemma 3.3, we have that

   $$c_{y,z} = d(y,z) = 2|\{\ell_j : \ell_j \in y \setminus z\}| = 2\sum_{j=1}^{p} v_{\ell_j,z}.$$

   (b) If $z \in V_{p+1}$, then

   $$c_{y,z} = d(y,z) = 2|\{\ell_j : \ell_j \in y \setminus z\}| + 1 = 2\sum_{j=1}^{p} v_{\ell_j,z} + 1.$$

   In both cases the third equality follows from the definition of v_ℓ.

   From (1a) and (1b) we have that

   $$C_y = 2\sum_{j=1}^{p} v_{\ell_j} + v_0.$$

2. Let $y = \{\ell_1, \ldots, \ell_{p+1}\} \in V_{p+1}$ with $\ell_1, \ldots, \ell_{p+1} \in \{1, \ldots, k\}$.

   (a) If $z \in V_p$, then

   $$c_{y,z} = d(y,z) = 2|\{\ell_j : \ell_j \in y \setminus z\}| - 1 = 2\sum_{j=1}^{p} v_{\ell_j,z} - 1,$$

   (b) If $z \in V_{p+1}$, then

   $$c_{y,z} = d(y,z) = 2|\{\ell_j : \ell_j \in y \setminus z\}| = 2\sum_{j=1}^{p+1} v_{\ell_j,z}.$$

So, we have

$$C_y = 2\sum_{j=1}^{p+1} v_{\ell_j} - v_0'.$$

Since v_0' is a linear combination of $v_0, \dots, v_k$, then C_y is a linear combination of $v_0, \dots, v_k$ as well.

Accordingly, we have proved that $\mathrm{Im}(D) \subseteq \mathrm{Span}(v_0, v_1, \dots, v_k)$. Hence, it follows that $\dim(\mathrm{Im}(D)) \leq k+1$, which implies that $\dim(\ker(D)) \geq \binom{k+1}{p+1} - k - 1$. □

In order to estimate the dimension of $\ker(D + 2\binom{k-1}{p}I)$, we need the following result.

Lemma 3.10. *The vectors $v_1, \dots v_k$ defined in Proposition* 3.9 *are linearly independent.*

Proof. Let $S := \sum_{\ell=1}^{k} \alpha_\ell \cdot v_\ell$ be a linear combination of the vectors $v_1, \dots, v_k$ with $\alpha_1, \dots, \alpha_k \in \mathbb{R}$ and assume that $S = 0$.

Let ℓ_0 be an element of $[k]$. Consider $z \in V_{p+1}$ such that $\ell_0 \in z$. The entry s_z of S is of the form

$$0 = s_z = \sum_{\ell=1}^{k} \alpha_\ell v_{\ell,z} = \sum_{\ell \in \{[k]-z\}} \alpha_\ell.$$

For $z' \in V_p$, where $z' = z \setminus \{\ell_0\}$, the entry $s_{z'}$ of S is

$$0 = s_{z'} = \sum_{\ell=1}^{k} \alpha_\ell v_{\ell,z'} = \left(\sum_{\ell \in \{[k]-z\}} \alpha_\ell \right) + \alpha_{\ell_0}.$$

Then

$$0 = s_{z'} - s_z = \alpha_{\ell_0}.$$

This implies that $\{v_1, \dots, v_k\}$ is a linear independent set. □

Proposition 3.11. *Let D be the distance matrix of the graph $G = G_{(k,p)}$. Then, for all $\ell \in \{1, \dots, k-1\}$ the vectors*

$$w_\ell := (w_{\ell,z})_{z \in V}, \qquad \textit{where} \qquad w_{\ell,z} := \begin{cases} k - |z|, & \textit{if } \ell \in z; \\ -|z|, & \textit{if } \ell \notin z, \end{cases}$$

belong to $\ker(D + 2\binom{k-1}{p}I)$ *and they are linearly independent. In particular,* $\dim(\ker(D + 2\binom{k-1}{p}))) \geq k-1$.

Proof. We begin by showing that the set $\{w_1, \dots, w_{k-1}\}$ is linearly independent. For all $\ell \in \{1, \dots, k-1\}$, $w_\ell = \sum_{j=1}^{k} v_j - k v_\ell$. In fact, given $z \in V$, if $\ell \in z$, then $v_{\ell,z} = 0$ holds

$$w_{\ell,z} = \sum_{j=1}^{k} v_{j,z} - k v_{\ell,z} = \sum_{j=1}^{k} v_{j,z} = k - |z|.$$

If $\ell \notin z$, then $v_{\ell,z} = \ell$ and

$$w_{\ell,z} = \sum_{j=1}^{k} v_{j,z} - kv_{\ell,z} = k - |z| - k = -|z|.$$

It follows that

$$\begin{aligned} w_1 &= (1-k)v_1 + v_2 + \ldots + v_k, \\ w_2 &= v_1 + (1-k)v_2 + \ldots + v_k, \\ &\vdots \\ w_{k-1} &= v_1 + v_2 + \ldots + (1-k)v_k. \end{aligned}$$

Let $\alpha_1, \ldots, \alpha_{k-1} \in \mathbb{R}$ be such that

$$\alpha_1 w_1 + \alpha_2 w_2 + \ldots \alpha_{k-1} w_{k-1} = 0.$$

By Lemma 3.10, and the fact that $v_1, \ldots, v_k$ are linearly independent, we obtained that

$$\beta_\ell = \alpha_1 + \ldots + \alpha_{\ell-1} + (1-k)\alpha_\ell + \alpha_{\ell+1} + \ldots \alpha_{k-1} = 0$$

for all $\ell \in \{1, \ldots, k-1\}$. If $1 \leq \ell \leq j \leq k-1$, then

$$0 = \beta_\ell - \beta_j = -k\alpha_\ell + k\alpha_j,$$

that is, $\alpha_\ell = \alpha_j$. Hence, $0 = \beta_\ell = (k-2)\alpha_\ell + (1-k)\alpha_\ell = -\alpha_\ell$. We conclude that the vectors $w_1, \ldots, w_{k-1}$ are linearly independent.

Claim 3.12. The vectors $w_1, \ldots, w_{k-1}$ belong to $\ker(D + 2\binom{k-1}{p} I)$.

The proof of this claim can be found in Appendix A and this conclude the proof of Proposition 3.11. □

Now, we are able to prove Theorem 3.1.

Proof of Theorem 3.1. Since P is an equitable partition, the eigenvalues of B are also eigenvalues of D. From Proposition 3.7, the eigenvalues of B are λ_1, 0, λ_2 and $-2\binom{k-1}{p}$. By Proposition 3.9, we know that 0 is an eigenvalue of D with multiplicity as least $\binom{k+1}{p+1} - k - 1$ and by Proposition 3.11, the multiplicity of $-2\binom{k-1}{p}$ is at least $k-1$. As D has order $\binom{k+1}{p+1}$, then

$$\begin{aligned} \binom{k+1}{p+1} &= m_D(0) + m_D\left(-2\binom{k-1}{p}\right) + m_D(\lambda_1) + m_D(\lambda_2) \\ &\geq \binom{k+1}{p+1} - k - 1 + k - 1 + m_D(\lambda_1) + m_D(\lambda_2). \end{aligned}$$

Since $m_D(\lambda_1), m_D(\lambda_2) \geq 1$, then all the inequalities above are in fact equalities

$$m_D(\lambda_1) = 1, m_D(0) = \binom{k+1}{p+1} - k - 1, m_D(\lambda_2) = 1, m_D\left(-2\binom{k-1}{p}\right) = k - 1,$$

and this finishes the proof of Theorem 3.1. □

Corollary 3.13. *With the notation as above, if $2p+1=k$ then the D-spectrum of the graph $G_{(k,p)}$ is contained in $\mathbb{Z}$. In fact, λ_1 and λ_2 are given by*

$$\lambda_1 = (2p+1)\binom{2p+1}{p} \qquad \textit{and} \qquad \lambda_2 = -\frac{\binom{2p}{p}}{p+1}.$$

Proof. From Theorem 3.1, it is enough to see that λ_1 and λ_2 are the roots of the polynomial

$$x^2 - 2p\binom{2p}{p}x - \binom{2p+1}{p}^2.$$

Furthermore, λ_1 and the sum $\lambda_1+\lambda_2$ are integers, so λ_2 is also an integer. □

Example 3.14. For the graph $G_{(6,2)}$ we will consider the vertices in the following order:

$\{1,2,3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,4\}, \{1,2,5\}, \{1,2,6\}, \{1,3,4\}, \{1,3,5\}, \{1,3,6\}, \{2,3,4\}, \{2,3,5\}, \{2,3,6\}, \{\{1,4\}, \{1,5\}, \{1,6\}, \{2,4\}, \{2,5\}, \{2,6\}, \{3,4\}, \{3,5\}, \{3,6\}, \{1,4,5\}, \{1,4,6\}, \{1,5,6\}, \{2,4,5\}, \{2,4,6\}, \{2,5,6\}, \{3,4,5\}, \{3,4,6\}, \{3,5,6\}, \{\{4,5\}, \{4,6\}, \{5,6\}, \{4,5,6\}$.

Fixing the vertex $x=\{1,2,3\}$ and taking the partition P as in (3.1), we obtain the quotient matrix B

$$B = \begin{pmatrix} 0 & 3 & 18 & 27 & 36 & 15 & 6 \\ 1 & 4 & 21 & 24 & 33 & 12 & 5 \\ 2 & 7 & 24 & 27 & 30 & 11 & 4 \\ 3 & 8 & 27 & 24 & 27 & 8 & 3 \\ 4 & 11 & 30 & 27 & 24 & 7 & 2 \\ 5 & 12 & 33 & 24 & 21 & 4 & 1 \\ 6 & 15 & 36 & 27 & 18 & 3 & 0 \end{pmatrix}. \tag{3.2}$$

As D has order $\binom{k+1}{p+1} = \binom{7}{3} = 35$, using the lower bounds for the multiplicities of the eigenvalues $0, 2\binom{k-1}{p} = -20, 50+5\sqrt{112}, 50-5\sqrt{112}$ of D obtained as described in Propositions 3.7, 3.9 and 3.11, we have that

$$\text{spect}_D(G) = \begin{bmatrix} -20 & 50-5\sqrt{112} & 0 & 50+5\sqrt{112} \\ \\ 5 & 1 & 28 & 1 \end{bmatrix},$$

by Theorem 3.1.

A Appendix

We use repeatedly Vandermonde's identity and the absorption formulas:

$$\sum_{i=0}^{r}\binom{n}{i}\binom{m}{r-i} = \binom{n+m}{r}, \quad \binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}, \quad \binom{n}{r} = \frac{n-r+1}{r}\binom{n}{r-1}.$$

Proof of Claim 3.8

Let us compute the coefficients of the matrix $\overline{B}$ defined by Equation (A.1). Recall that we are always considering the case where $2p+1 \le k$.

$\overline{B} := B^t|_{\mathrm{Span}(L_0,F_0,F_1)}$ can be written in the basis $\{L_0, F_0, F_1\}$ as

$$\overline{B} = \begin{pmatrix} \alpha(L_0) & \alpha(F_0) & \alpha(F_1) \\ \beta(L_0) & \beta(F_0) & \beta(F_1) \\ \delta(L_0) & \delta(F_0) & \delta(F_1) \end{pmatrix} \tag{A.1}$$

We need to compute the functions α, β and δ calculated in L_0, F_0 and F_1 where

$$\alpha(v) := \textstyle\sum_{j=0}^{d} v_j = \sum_{s=0}^{\lfloor \frac{d}{2} \rfloor} v_{2s} + \sum_{s=1}^{p+1} v_{2s-1}, \qquad \beta(v) := \sum_{s=0}^{\lfloor \frac{d}{2} \rfloor} s v_{2s} + \sum_{s=1}^{p+1} s v_{2s-1},$$
$$\delta(v) := \textstyle\sum_{s=0}^{\lfloor \frac{d}{2} \rfloor} s v_{0,2s} + \sum_{s=1}^{p+1} (s-1) v_{2s-1}.$$

Remember that $L_0 = (L_{0,j})_{j \in \{0,\ldots,d\}}$ is given by

$$L_{0,j} := b_{0j} = \begin{cases} 2(p+1)\binom{p+1}{p+1-s}\binom{k-p-1}{s} - 2(p+1)\binom{k-p-1}{s}\binom{p}{p-s} & \text{if } j = 2s, \\ (2p+1)\binom{p+1}{p+1-s}\binom{k-p-1}{s-1} - 2(p+1)\binom{k-p-1}{s-1}\binom{p}{p-s} & \text{if } j = 2s-1. \end{cases}$$

Remark A.1. Note that, if $k = 2p+1$, then $\lfloor \frac{d}{2} \rfloor = p$. As $\binom{k-p-1}{p+1} = 0$, we can assume that $L_{0,2p+2} := 0$. The same holds for L_0 and L_1. This will simplify the expressions of the sums below.

$$\sum_{s=0}^{p+1} L_{0,2s} = 2(p+1)\left(\binom{k}{p+1} - \binom{k-1}{p}\right) = 2(k-p-1)\binom{k-1}{p},$$

$$\sum_{s=1}^{p+1} L_{0,2s-1} = (2p+1)\binom{k}{p} - 2(p+1)\binom{k-1}{p-1} = \binom{k-1}{p}\left(\frac{2kp-2p^2+k-2p}{k-p}\right).$$

$$\sum_{s=0}^{p+1} sL_{0,2s} = 2(p+1)(k-p-1)\left(\binom{k-1}{p} - \binom{k-2}{p-1}\right) = \binom{k-1}{p}\frac{2(p+1)(k-p-1)^2}{k-1},$$

$$\sum_{s=1}^{p+1} sL_{0,2s-1} = (2p+1)(p+1)\binom{k-1}{p} - 2(p+1)p\binom{k-2}{p-1},$$

$$= \binom{k-1}{p}\frac{(p+1)(2kp-p^2+k-2p-1)}{k-1},$$

$$\sum_{s=1}^{p+1}(s-1)L_{0,2s-1} = (k-p-1)\left((2p+1)\binom{k-1}{p-1} - 2(p+1)\binom{k-2}{p-2}\right)$$

$$= \binom{k-1}{p}\frac{p(k-p-1)(2kp-2p^2+k-2p-1}{(k-1)(k-p)}.$$

Finally we obtain:

$$\begin{aligned}
\alpha(L_0) =&\binom{k-1}{p}\left(\frac{k(2k-2p-1)}{k-p}\right),\\
\beta(L_0) =&\binom{k-1}{p}(p+1)(2k-2p-1),\\
\delta(L_0) =&\binom{k-1}{p}(k-p-1)\left(\frac{2pk-2p^2+2k-3p}{k-p}\right).
\end{aligned}$$

Analogously, as F_0 is given by

$$F_{0,j} := \begin{cases} -\binom{p+1}{p+1-s}\binom{k-p-1}{s}+2\binom{k-p-1}{s}\binom{p}{p-s} & \text{if } i=2s,\\ -\binom{p+1}{p+1-s}\binom{k-p-1}{s-1}+2\binom{k-p-1}{s-1}\binom{p}{p-s} & \text{if } i=2s-1. \end{cases}$$

we have

$$\begin{aligned}
\alpha(F_0) =&-\binom{k-1}{p}\frac{k-2p-2}{p+1}-\binom{k-1}{p}\frac{k-2p}{k-p}\\
=&-\binom{k-1}{p}\frac{k(k-2p-1)}{(k-p)(p+1)},\\
\beta(F_0) =&-\binom{k-1}{p}\frac{(k-p-1)(k-2p-1)}{k-1}-\binom{k-1}{p}\frac{kp-2p^2+k-p-1}{k-1}\\
=&(2p-k)\binom{k-1}{p},\\
\delta(F_0) =&-\binom{k-1}{p}\frac{(k-p-1)(k-2p-1)}{k-1}-\binom{k-1}{p}\frac{p(k-p-1)(k-2p-1}{(k-p)(k-1)}\\
=&\binom{k-1}{p}\frac{(k-p-1)(2p-k)}{(k-p)}.
\end{aligned}$$

Finally as F_1 is given by

$$F_{1,j} := \begin{cases} \binom{p+1}{p+1-s}\binom{k-p-1}{s}-2\binom{p+1}{p+1-s}\binom{k-p-2}{s-1} & \text{if } i=2s,\\ \binom{p+1}{p+1-s}\binom{k-p-1}{s-1}-2\binom{p+1}{p+1-s}\binom{k-p-2}{s-2} & \text{if } i=2s-1, \end{cases}$$

direct computations yield

$$\begin{aligned}
\alpha(F_1) =&\binom{k-1}{p}\frac{k-2p-2}{p+1}+\binom{k-1}{p}\frac{k-2p}{k-p}\\
=&\binom{k-1}{p}\frac{k(k-2p-1)}{(k-p)(p+1)},\\
\beta(F_1) =&\binom{k-1}{p}\frac{(k-p-1)(k-2p-3)}{(k-1)}+\binom{k-1}{p}\frac{(p+1)(k-2p-1)}{k-1}\\
=&(k-2p-2)\binom{k-1}{p},\\
\delta(F_1) =&\binom{k-1}{p}\frac{(k-p-1)(k-2p-3)}{(k-1)}+\binom{k-1}{p}\frac{p(k^2-3kp+2p^2-2k+3p-1)}{(k-1)(k-p)}\\
=&\binom{k-1}{p}\frac{k^2-3kp+2p^2-3k+4p}{k-p}.
\end{aligned}$$

Now, the entries of the matrix $\overline{B}$ are completely calculated.

Proof of Claim 3.12

Let us begin with a lemma.

Lemma A.2. *Let W be a set of order n, $i \in W$ and $U \subseteq W$ such that $|U| = m$. For all $0 \leq \overline{r} \leq n$, the following hold.*

1. *If $i \in U$, then*

$$\sum_{\substack{A\in[W,\overline{r}]\\ i\in A}} |A\cap U| = \binom{n-1}{\overline{r}-1}+(m-1)\binom{n-2}{\overline{r}-2}, \qquad \sum_{\substack{A\in[W,\overline{r}]\\ i\notin A}} |A\cap U| = (m-1)\binom{n-2}{\overline{r}-1}.$$

2. *If $i \notin U$, then*

$$\sum_{\substack{A\in[W,\overline{r}]\\ i\in A}} |A\cap U| = m\binom{n-2}{\overline{r}-2}, \qquad \sum_{\substack{A\in[W,\overline{r}]\\ i\notin A}} |A\cap U| = m\binom{n-2}{\overline{r}-1}.$$

The proof is straightforward by Lemma 3.4.

Now we can complete the proof of Proposition 3.11.

We need to prove that the vectors $w_1, \dots, w_{k-1}$ belong to $\ker(D + 2\binom{k-1}{p}I)$, that is, we need to show that for all $i \in \{1, \dots, k-1\}$, $(D + 2\binom{k-1}{p}I)w_i = 0$ holds. If $L_y = (L_{y,z})_{z\in V}$ is the row vector of the matrix $D + 2\binom{k-1}{p}I$ indexed by $y \in V$, we have that

$$L_{y,z} = \begin{cases} d(y,z) & \text{if} \quad y \neq z,\\ d(y,z)+2\binom{k-1}{p} & \text{if} \quad y = z.\end{cases}$$

Then, it is sufficient to prove that

$$\langle L_y, w_i\rangle = \sum_{z\in V} L_{y,z}\cdot w_{i,z} = 0.$$

Observe that

$$\begin{aligned}\sum_{z\in V} L_{y,z}\cdot w_{i,z} &= \sum_{\substack{z\in V_{p+1}\\ i\in z}} d(y,z)\cdot w_{i,z} + \sum_{\substack{z\in V_{p+1}\\ i\notin z}} d(y,z)\cdot w_{i,z} + \sum_{\substack{z\in V_{p}\\ i\in z}} d(y,z)\cdot w_{i,z}\\ &\quad + \sum_{\substack{z\in V_{p}\\ i\notin z}} d(y,z)\cdot w_{i,z} + 2\binom{k-1}{p}\cdot w_{i,y}\\ &= \sum_{\substack{z\in V_{p+1}\\ i\in z}} d(y,z)\cdot (k-p-1) + \sum_{\substack{z\in V_{p+1}\\ i\notin z}} d(y,z)\cdot(-p-1)\\ &\quad + \sum_{\substack{z\in V_{p}\\ i\in z}} d(y,z)\cdot (k-p) + \sum_{\substack{z\in V_{p}\\ i\notin z}} d(y,z)\cdot(-p) + 2\binom{k-1}{p}\cdot w_{i,y}.\end{aligned} \tag{A.2}$$

We analyze four cases: $i\in y\in V_p$, $i\notin y\in V_p$, $i\in y\in V_{p+1}$ and $i\notin y\in V_{p+1}$. For the first case, as $y\in V_p$ and $i\in y$, from Lemmas 3.3 and A.2, we have

$$\begin{aligned}\sum_{\substack{z\in V_{p+1}\\ i\in z}} d(y,z) &= (2p-1)\binom{k-1}{p} - 2(p-1)\binom{k-2}{p-1}\\ \sum_{\substack{z\in V_{p+1}\\ i\notin z}} d(y,z) &= (2p+1)\binom{k-1}{p+1} - 2(p-1)\binom{k-2}{p}\\ \sum_{\substack{z\in V_{p}\\ i\in z}} d(y,z) &= (2p-2)\binom{k-1}{p-1} - 2(p-1)\binom{k-2}{p-2}\\ \sum_{\substack{z\in V_{p}\\ i\notin z}} d(y,z) &= 2p\binom{k-1}{p} - 2(p-1)\binom{k-2}{p-1}\end{aligned}$$

Moreover, as $w_{i,y}=k-p$, it follows that

$$\begin{aligned}\frac{\langle L_y, w_i\rangle}{\binom{k-1}{p}} &= (2p-1)(k-p-1) - \frac{2p(p-1)(k-p-1)}{k-1} - (2p+1)(k-p-1)\\ &\quad + \frac{2(p-1)(p+1)(k-p-1)}{k-1} + 2p(p-1) - \frac{2p(p-1)^2}{k-1} - 2p^2\\ &\quad + \frac{2p^2(p-1)}{k-1} + 2(k-p) = 0.\end{aligned}$$

This ends the first case. The others cases are analogous. Now the proof of Proposition 3.11 is completed.

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