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A necessary and sufficient spectral condition for a tree to have a perfect matching

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Abstract

For a tree T on even n vertices, we introduce a new $n \times n$ matrix, called the fictitious matrix of T, which is denoted by F. We prove that T has a perfect matching if and only if $\det(F) = \pm 1$. Using the eigenvalues of F, we also present a necessary and sufficient condition for T to have a perfect matching.

1 Introduction

Let G be a simple connected graph on n vertices $1, 2, \ldots, n$ and m edges $e_1, e_2, \ldots, e_m$ with the adjacency matrix A and the degree matrix D. The *signless Laplacian* Q of G is defined as $Q = D + A$. The vertex-edge *incidence matrix* M of G is the $n \times m$ matrix whose (i, j)-entry is 1 if vertex i is incident with edge e_j and 0 otherwise. We introduce a new $n \times n$ matrix F for G, called the *fictitious matrix* of G, whose (i, j)-entry is the product of the (i, j)-entry of Q and $(-1)^{d(i,j)}$ where $d(i, j)$ is the distance between vertices i and j in G.

A *matching* in G is a set of edges such that no two edges have a common vertex. A *perfect matching* in G on n vertices, n being even, is a matching consisting of $\frac{n}{2}$ edges. There are several known necessary and sufficient conditions for a graph to have a perfect matching [3]. But there are no spectral conditions for a graph to have a perfect matching. In this article, we investigate spectral conditions for a tree to have a perfect matching. In section 2, we find the inverse of the fictitious matrix of a tree. In section 3, we study the connection between perfect matchings in a tree and the fictitious matrix of the tree.

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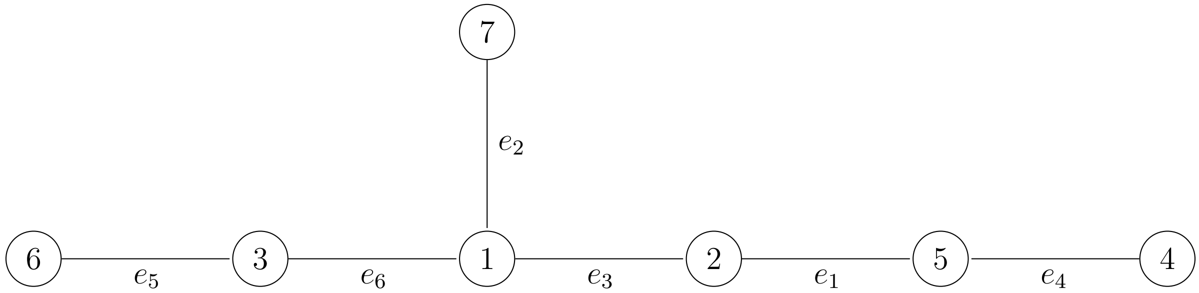

Figure 1: The smallest asymmetric tree

2 The inverse of a fictitious matrix

First we note the following result for the signless Laplacian of a connected graph.

Theorem 2.1. *[2, Theorem 2.1] The smallest eigenvalue of the signless Laplacian of a connected graph is equal to* 0 *if and only if the graph is bipartite. In this case* 0 *is a simple eigenvalue.*

Now we consider an $n \times n$ matrix H whose rows and columns are indexed by the vertices $1, 2, \ldots, n$ of a tree T and $H = [h_{i,j}]$ is defined in [1, page 901] as follows:

$$h_{i,j} = \frac{(-1)^{d(i,j)}}{n} \begin{cases} 1 & \text{if } i \leq j \\ -1 & \text{if } i > j. \end{cases} \tag{2.1}$$

Example 2.2. For the tree given in Figure 1,

$$H = \frac{1}{7} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

Theorem 2.3. *Let T be a tree on n vertices with the fictitious matrix F. For the matrix H defined in (2.1), we have $HF = I_n$.*

Proof. Let $F = [f_{i,j}]$. Suppose $i, j \in \{1, \ldots, n\}$. Then the (i,j)-entry of HF is given by

$$(HF)_{i,j} = \sum_{k=1}^{n} h_{i,k} f_{k,j}.$$

Case 1. $i = j$

$$(HF)_{i,i} = \sum_{k=1}^{n} h_{i,k} f_{k,i} = \text{do math here} = 0$$

Case 2. $i \neq j$
Without loss of generality, let $i < j$.

$$\begin{aligned}(HF)_{i,j} &= \sum_{k=1}^{n} h_{i,k} f_{k,j} \\ &= \sum_{k=1}^{i-1} h_{i,k} f_{k,j} + \sum_{k=i}^{j-1} h_{i,k} f_{k,j} + \sum_{k=j}^{n} h_{i,k} f_{k,j} \\ &= \text{do math here} \\ &= 1.\end{aligned}$$

Thus $HF = I_n$. □

Corollary 2.4. *The fictitious matrix of a tree is invertible.*

3 Fictitious matrix and perfect matchings

In this section we study the connection between perfect matchings in a tree and the fictitious matrix of the tree.

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