A necessary and sufficient spectral condition for a tree to have a perfect matching

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Abstract

For a tree \( T \) on even \( n \) vertices, we introduce a new \( n \times n \) matrix, called the fictitious matrix of \( T \), which is denoted by \( F \). We prove that \( T \) has a perfect matching if and only if \( \det(F) = \pm 1 \). Using the eigenvalues of \( F \), we also present a necessary and sufficient condition for \( T \) to have a perfect matching.

1 Introduction

Let \( G \) be a simple connected graph on \( n \) vertices \( 1, 2, \ldots, n \) and \( m \) edges \( e_1, e_2, \ldots, e_m \) with the adjacency matrix \( A \) and the degree matrix \( D \). The signless Laplacian \( Q \) of \( G \) is defined as \( Q = D + A \). The vertex-edge incidence matrix \( M \) of \( G \) is the \( n \times m \) matrix whose \((i, j)\)-entry is 1 if vertex \( i \) is incident with edge \( e_j \) and 0 otherwise. We introduce a new \( n \times n \) matrix \( F \) for \( G \), called the fictitious matrix of \( G \), whose \((i, j)\)-entry is the product of the \((i, j)\)-entry of \( Q \) and \((-1)^{d(i,j)}\) where \( d(i, j) \) is the distance between vertices \( i \) and \( j \) in \( G \).

A matching in \( G \) is a set of edges such that no two edges have a common vertex. A perfect matching in \( G \) on \( n \) vertices, \( n \) being even, is a matching consisting of \( \frac{n}{2} \) edges. There are several known necessary and sufficient conditions for a graph to have a perfect matching [3]. But there are no spectral conditions for a graph to have a perfect matching. In this article, we investigate spectral conditions for a tree to have a perfect matching. In section 2, we find the inverse of the fictitious matrix of a tree. In section 3, we study the connection between perfect matchings in a tree and the fictitious matrix of the tree.

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2 The inverse of a fictitious matrix

First we note the following result for the signless Laplacian of a connected graph.

**Theorem 2.1.** [2, Theorem 2.1] The smallest eigenvalue of the signless Laplacian of a connected graph is equal to 0 if and only if the graph is bipartite. In this case 0 is a simple eigenvalue.

Now we consider an $n \times n$ matrix $H$ whose rows and columns are indexed by the vertices $1, 2, \ldots, n$ of a tree $T$ and $H = [h_{i,j}]$ is defined in [1, page 901] as follows:

$$h_{i,j} = \frac{(-1)^{d(i,j)}}{n} \begin{cases} 1 & \text{if } i \leq j \\ -1 & \text{if } i > j. \end{cases} \quad (2.1)$$

**Example 2.2.** For the tree given in Figure 1,

$$H = \frac{1}{7} \begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

**Theorem 2.3.** Let $T$ be a tree on $n$ vertices with the fictitious matrix $F$. For the matrix $H$ defined in (2.1), we have $HF = I_n$.

**Proof.** Let $F = [f_{i,j}]$. Suppose $i, j \in \{1, \ldots, n\}$. Then the $(i, j)$-entry of $HF$ is given by

$$(HF)_{i,j} = \sum_{k=1}^{n} h_{i,k} f_{k,j}.$$ 

**Case 1.** $i = j$

$$(HF)_{i,i} = \sum_{k=1}^{n} h_{i,k} f_{k,i} = \text{do math here} = 0$$
Case 2. \( i \neq j \)
Without loss of generality, let \( i < j \).

\[
(HF)_{i,j} = \sum_{k=1}^{n} h_{i,k}f_{k,j} \\
= \sum_{k=1}^{i-1} h_{i,k}f_{k,j} + \sum_{k=i}^{j-1} h_{i,k}f_{k,j} + \sum_{k=j}^{n} h_{i,k}f_{k,j} \\
= \text{do math here} \\
= 1.
\]

Thus \( HF = I_n \). \( \square \)

**Corollary 2.4.** The fictitious matrix of a tree is invertible.

### 3 Fictitious matrix and perfect matchings

In this section we study the connection between perfect matchings in a tree and the fictitious matrix of the tree.

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**References**


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