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Research Article

The nullity of the net Laplacian matrix of a signed graph

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Abstract

The net Laplacian matrix of a signed graph $\Gamma = (G, \sigma)$, where $G = (V(G), E(G))$ is an unsigned graph (referred to as the underlying graph) and $\sigma : E(G) \to \{-1, +1\}$ is the sign function, is defined as $L^{\pm}(\Gamma) = D^{\pm}(\Gamma) - A(\Gamma)$. Here, $D^{\pm}(\Gamma)$ and $A(\Gamma)$ represent the diagonal matrix of net-degrees and the adjacency matrix of Γ, respectively. The nullity of $L^{\pm}(\Gamma)$, denoted as $\eta(L^{\pm}(\Gamma))$, refers to the multiplicity of 0 as an eigenvalue of $L^{\pm}(\Gamma)$. In this paper, we concentrate on the nullity of the net Laplacian matrix of a connected signed graph Γ, and establish that $1 \le \eta(L^{\pm}(\Gamma)) \le \min\{\beta(\Gamma)+1, |V(\Gamma)|-1\}$, where $\beta(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$ denotes the cyclomatic number of Γ. We completely determine the connected signed graphs with nullity $|V(\Gamma)| - 1$. Additionally, we characterize the signed cactus graphs with nullity 1 or $\beta(\Gamma) + 1$.

1 Introduction

A signed graph Γ of order n is a pair (G, σ), where $G = (V(G), E(G))$ is an unsigned graph with vertex set $V(G)$ and edge set $E(G)$, referred to as the underlying graph, and $\sigma : E(G) \to \{-1, +1\}$ is the sign function. For a vertex v of Γ, the positive degree $d^+_\Gamma(v)$ of v in Γ is the number of positive neighbors of v (i.e., those adjacent to v by a positive edge). In the similar way, we define the negative degree $d^-_\Gamma(v)$. The net-degree of v in Γ is defined as $d^{\pm}_\Gamma(v) = d^+_\Gamma(v) - d^-_\Gamma(v)$.

Given a matrix M, the spectrum of M is denoted by $Spec(M) = \{\lambda_1(M)^{k_1}, \cdots, \lambda_i(M)^{k_i}\}$, where the superscripts denote the multiplicities of corresponding eigenvalues. Throught this paper, the eigenvalues of any matrix are arranged in non-increasing order. The rank and nullity of M are denoted by $r(M)$ and $\eta(M)$, respectively. The adjacency matrix $A(\Gamma)$ of a signed graph Γ is obtained from the adjacency matrix of the underlying graph by reversing the sign of all 1s corresponding to negative edges. The net Laplacian matrix of Γ is defined as $L^{\pm}(\Gamma) = D^{\pm}(\Gamma) - A(\Gamma)$, where $D^{\pm}(\Gamma)$ is the diagonal matrix of net-degrees.

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The topic of the nullity of the adjacency matrix of a simple graph has garnered significant attention recently. Collatz and Sinogowitz [8] originally posed the problem of characterizing all singular graphs. The nullity of a graph holds a prominent place in spectral graph theory due to its applications in chemistry. In the Hückel molecular orbital model, if $\eta(A(G)) > 0$ for the molecular graph G, then the corresponding chemical compound is highly reactive and unstable, or may even be nonexistent (refer to [1] or [9]). In the study of this problem, researchers have focused on bounding the nullity of a graph by utilizing various structural parameters, such as the order, the maximum degree, the number of pendent vertices, and the graph's cyclomatic number, etc (see, for example, [4, 5, 6, 9, 21, 26, 27]).

The nullity of the adjacency matrix of a signed graph also has been widely studied (see [7, 12, 13, 18, 19, 23] and reference therein). This problem is closely related to the minimum rank problem of symmetric matrices whose patterns are described by graphs [11]. Here, we consider this problem with respect to the net Laplacian matrix. The significance of the spectrum of the net Laplacian matrix in control theory was recognized in [15]. The same topic is studied in [24] from a graph theoretic insight. The advantages of use of the net Laplacian matrix instead of the Laplacian matrix (in study of signed graphs) is investigated in [25]. Very recently, Mallik [20] introduced a new oriented incidence matrix of a signed graph, by which the matrix tree theorem of the net Laplacian matrix of a signed graph is given. In this paper, we investigate the nullity of the net Laplacian matrix of a connected signed graph, which relies on the study of the characteristic polynomial of the net Laplacain matrix of this signed graph. In 1982, Chaiken [3] gave a combinatorial proof of the all minors matrix tree theorem. In 2016, Buslov [2] proposed an alternative proof based on the straightforward computation of the minors of incidence matrices and on revealing a connection of them with forests. The above two papers established a way for computing any coefficient of the characteristic polynomial of the Laplace matrix of a weighted digraph. Here, we rewrite it in the form of the net Laplacian matrix of a signed graph Γ. The proof can be obtained directly from [3, All minors matrix tree theorem] or [2, Theorem 2]. Denote by $\mathcal{F}^k(\Gamma)$ the set of all spanning k-component forests of Γ. For $F^k(\Gamma) \in \mathcal{F}^k(\Gamma)$, $a(F^k(\Gamma)) = n_1 \dots n_k$, where n_i is the number of the vertices of i-component of $F^k(\Gamma)$.

Theorem 1.1. *Let*

$$P_{L^{\pm}(\Gamma)}(x) = \det(xI - L^{\pm}(\Gamma)) = \sum_{k=0}^{n} c_k x^k$$

be the characteristic polynomial of the net Laplacian matrix of a signed graph Γ. *Then*

$$c_k = (-1)^{n-k}\Big(\sum_{F^k(\Gamma)\in\mathcal{F}^k(\Gamma)} a(F^k(\Gamma))\sigma(F^k(\Gamma))\Big).$$

For simplicity, we will henceforth refer to the spectrum, nullity, and rank of $L^{\pm}(\Gamma)$ as the spectrum, nullity, and rank of Γ, denoted by $Spec(\Gamma)$, $\eta(\Gamma)$, and $r(\Gamma)$, respectively. It is clear that for a signed graph of order n, we have $r(\Gamma) + \eta(\Gamma) = n$. Notably, $L^{\pm}(\Gamma)$ is a symmetric matrix, and the sum of entries in each row is zero. Thus, $\eta(\Gamma) \geq 1$ holds for any signed graph Γ, and $\eta(\Gamma) = 1$ if and only if $c_1 \neq 0$, where the coefficient c_1 of the linear term of $P_{L^{\pm}(\Gamma)}(x)$ given by

$$(-1)^{n-1}(\sum_{F^1(\Gamma)\in\mathcal{F}^1(\Gamma)} a(F^1(\Gamma))\sigma(F^1(\Gamma))) = (-1)^{n-1}n(\sum_{F^1(\Gamma)\in\mathcal{F}^1(\Gamma)} \sigma(F^1(\Gamma))).$$

The cyclomatic number of a connected signed graph Γ, denoted by $\beta(\Gamma)$, is defined as $\beta(\Gamma) = |E(\Gamma)| - |V(\Gamma)| + 1$. A connected signed graph Γ is called a signed tree if $\beta(\Gamma) = 0$, a signed unicyclic graph if $\beta(\Gamma) = 1$, and a signed bicyclic graph if $\beta(\Gamma) = 2$.

It should be noted that $\eta(\Gamma) = \eta(-\Gamma)$, where $-\Gamma$ is obtained by reversing the sign of each edge in Γ. A signed cactus graph Γ is a signed graph whose underlying graph is a cactus graph. Recall that a cactus graph is a connected graph in which any two cycles have no edge in common. Equivalently, it is a connected graph in which any two cycles have most one vertex in common. With this background, we present the main result.

Theorem 1.2. *Let Γ be a connected signed graph of order n ($n \geq 2$) with cyclomatic number $\beta(\Gamma)$. Then*

(i) $1 \leq \eta(\Gamma) \leq min\{\beta(\Gamma) + 1, n - 1\}$,

(ii) $\eta(\Gamma) = n - 1$ if and only if n is an even number and $\Gamma = K_{\frac{n}{2}}\nabla^- K_{\frac{n}{2}}$ or $-(K_{\frac{n}{2}}\nabla^- K_{\frac{n}{2}})$, where $K_{\frac{n}{2}}$ is the $\frac{n}{2}$-vertices signed complete graph with all positive edges and $K_{\frac{n}{2}}\nabla^- K_{\frac{n}{2}}$ is obtained by adding all possible negative edges between vertices of one $K_{\frac{n}{2}}$ and vertices of another one.

Moreover, if Γ is a signed cactus graph, then

(iii) $\eta(\Gamma) = 1$ if and only if $m^+(C) \neq m^-(C)$ for any cycle C of Γ,

(iv) $\eta(\Gamma) = \beta(\Gamma) + 1$ if and only if $m^+(C) = m^-(C)$ for any cycle C of Γ, where $m^+(C)$ and $m^-(C)$ are the numbers of the positive and negative edges of C, respectively.

The inequalities $1 \leq \eta(\Gamma) \leq \beta(\Gamma) + 1$ have also been established by Ge and Liu in [16, Theorem 6.17]. In this paper, we provide a concise proof of these inequalities. Additionally, as a by-product of the study on the spectra of signed complete graphs, Ou, Hou, and Xiong proved in [22, Corollay 2.9] that, for an n-vertices signed complete graph (K_n, σ), $\eta((K_n, \sigma)) = n - 1$ if and only if (K_n, σ) is $K_{\frac{n}{2}}\nabla^- K_{\frac{n}{2}}$ or $-(K_{\frac{n}{2}}\nabla^- K_{\frac{n}{2}})$. Here, without computing the spectra, we extend this result from signed complete graphs to connected signed graphs.

The remainder of this paper is organized as follows. Section 2 introduces some lemmas. In Section 3, we provide the proof of Theorem 1.2. Section 4 offers remarks and discussions on our results.

2 Preliminaries

We first recall some notations not defined in Section 1. Let Γ be a signed graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. A subgraph H of Γ is a signed graph such that $V(H) \subseteq V(\Gamma)$, $E(H) \subseteq E(\Gamma)$ and the edge set $E(H)$ preserving the signs in Γ. Furthermore, H is called an induced subgraph of Γ if for all $u, v \in V(H)$, u, v are adjacent in H if and only if they are adjacent in Γ. The sign of a subgraph H of Γ is defined as $\sigma(H) = \prod_{e\in E(H)} \sigma(e)$ and the numbers of positive and negative edges of H are denoted by $m^+(H)$ and $m^-(H)$, respectively. If $V_1 \subseteq V(\Gamma)$, we denote by $\Gamma[V_1]$ the induced subgraph of Γ with vertex set V_1, and denote by $\Gamma - V_1$ the induced subgraph of Γ with vertex set $V(\Gamma) \setminus V_1$, i.e., $\Gamma - V_1 = \Gamma[V(\Gamma) \setminus V_1]$. We simplify $\Gamma - V_1$ as $\Gamma - v$ when $V_1 = \{v\}$. For an induced subgraph H of Γ and a vertex subset $V_1 \subset V(\Gamma)$ outside H, denote by $H + V_1$ the induced subgraph of Γ with vertex set $V(H) \cup V_1$. Sometimes we use the notation $\Gamma - H$ instead of $\Gamma - V(H)$ if H is an induced subgraph of Γ. For an edge subset $E_1 \subseteq E(\Gamma)$, we denote by $\Gamma - E_1$ the signed graph with the same vertex set as Γ and with edge set $E(\Gamma) \setminus E_1$. We also abbreviate $\Gamma - E_1$ as $\Gamma - e$ when $E_1 = \{e\}$. An edge e (resp., a vertex v) is called a cut edge (resp., a cut vertex) if $\Gamma - e$ (resp., $\Gamma - v$) has more connected components than Γ. A signed graph Γ with a cut vertex w can be regard as a coalescence $\Gamma_1 \cdot \Gamma_2$ of two signed graphs Γ_1 and Γ_2, obtained from $\Gamma_1 \cup \Gamma_2$ by identifying a vertex u of Γ_1 with a vertex v of Γ_2. Formally, $V(\Gamma_1 \cdot \Gamma_2) = V(\Gamma_1 - u) \cup V(\Gamma_2 - v) \cup \{w\}$ with two vertices in $\Gamma_1 \cdot \Gamma_2$ adjacent if they are adjacent in Γ_1 or Γ_2, or if one is w and the other is a neighbor of u in Γ_1 or a neighbor of v in Γ_2.

Next we present some preliminary results that will be useful later on. The following lemma provides fundamental properties of the nullity of a signed graph.

Lemma 2.1. *Let Γ be a signed graph of order n.*

(1) *If $\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_t$, where $\Gamma_1, \cdots, \Gamma_t$ are all the connected components of Γ, then $\eta(\Gamma) = \sum_{i=1}^{t} \eta(\Gamma_i)$.*

(2) *$\eta(\Gamma) = n$ if and only if Γ has no edges.*

We now introduce an analogue of the Interlacing Theorem (cf. [17, Theorem 2.1]) for the net Laplacian matrix of a signed graph with respect to edges. For this purpose, we need the following lemma, known as the Courant-Weyl inequalities.

Lemma 2.2. *[10, Theorem 1.3.15] Let A and B be $n \times n$ Hermitian matrices. Then*

$$\lambda_i(A+B) \le \lambda_j(A) + \lambda_{i-j+1}(B) \qquad (1 \le j \le i \le n),$$
$$\lambda_i(A+B) \ge \lambda_j(A) + \lambda_{i-j+n}(B) \qquad (1 \le i \le j \le n).$$

Note that we cannot invoke an analogue for the Interlacing Theorem of the net Laplacian matrix when we delete vertices because a principal submatrix of $L^{\pm}$ is not the net Laplacian matrix of the corresponding induced subgraph. However, we do have an analogue for the Interlacing Theorem when we delete an edge:

Lemma 2.3. *Let Γ be a signed graph of order n. If $e = uv$ is an edge of Γ and $H = \Gamma - e$, then*

$$\lambda_1(\Gamma) \ge \lambda_1(H) \ge \cdots \ge \lambda_n(\Gamma) \ge \lambda_n(H), \qquad if\ \sigma(e) = +1.$$
$$\lambda_1(H) \ge \lambda_1(\Gamma) \ge \cdots \ge \lambda_n(H) \ge \lambda_n(\Gamma), \qquad if\ \sigma(e) = -1.$$

Proof. We can write $L^{\pm}(\Gamma)$ as $L^{\pm}(H) + Q$, where

$$Q = \begin{array}{c c} & \begin{array}{ccc} u & v & \end{array} \\ \begin{array}{c} u \\ v \\ \\ \end{array} & \begin{bmatrix} \sigma(e) & -\sigma(e) & \mathbf{0}_{1\times(n-2)} \\ -\sigma(e) & \sigma(e) & \mathbf{0}_{1\times(n-2)} \\ \mathbf{0}_{(n-2)\times 1} & \mathbf{0}_{(n-2)\times 1} & \mathbf{0}_{(n-2)\times(n-2)} \end{bmatrix} \end{array}.$$

Note that the spectrum of Q is $\{2^1, 0^{n-1}\}$ if $\sigma(e) = +1$, and is $\{0^{n-1}, (-2)^1\}$ if $\sigma(e) = -1$. Then by Lemma 2.2, we obtain the result. $\square$

We conclude this section with a direct consequence of Lemma 2.3.

Corollary 2.4. *Let Γ be a signed graph. If we delete an edge e of Γ, then $\eta(\Gamma) - 1 \leq \eta(\Gamma - e) \leq \eta(\Gamma) + 1$.*

3 Proof of Theorem 1.2

This section is dedicated to proving Theorem 1.2. We first present two helpful lemmas as preparatory steps, focusing on the nullities of signed trees and signed unicyclic graphs. In [14, Theorem 2.2], it was proven that for an n-vertices signed tree T, the numbers of positive, negative, and zero eigenvalues of T are $m^+(T)$, $m^-(T)$, and $n - m^+(T) - m^-(T) = 1$, respectively. Here, we also provide an alternative proof for self-containment.

Lemma 3.1. *Let T be a signed tree of order n. Then $\eta(T) = 1$.*

Proof. By Theorem 1.1 and $\eta(T) \geq 1$, it is sufficient to prove $c_1 \neq 0$ in the case $\Gamma = T$. The result follows from $c_1 = (-1)^{n-1} n \sum_{F^1(T) \in \mathcal{F}^1(T)} \sigma(F^1(T)) = (-1)^{n-1} n \cdot \sigma(T) \neq 0$. $\square$

Lemma 3.2. *Let U be a signed unicyclic graph of order n with the unique signed cycle C. Then*

$$\eta(U) = \begin{cases} 1, & if\ m^+(C) \neq m^-(C), \\ 2, & otherwise. \end{cases}$$

Proof. As previously shown, we need to prove that, in the case $\Gamma = U$, $c_1 \neq 0$ if $m^+(C) \neq m^-(C)$ and $c_1 = 0$ otherwise. Since

$$\sum_{F^1(U) \in \mathcal{F}^1(U)} \sigma(F^1(U)) = \sum_{e \in E(C)} \sigma(U - e) = \sigma(U) \sum_{e \in E(C)} \sigma(e),$$

where the first equality follows from the fact that an 1-component spanning forest (i.e., a spanning tree) of U is obtained by deleting an edge of C, so we have $c_1 \neq 0$ and $\eta(U) = 1$ if $m^+(C) \neq m^-(C)$. If $m^+(C) = m^-(C)$, on the one hand, $c_1 = 0$ and so $\eta(U) \geq 2$. On the other hand, by Corollary 2.4 and Lemma 3.1 we have $\eta(U) \leq \eta(U - e) + 1 = 2$, where $e \in E(C)$. This completes the proof of Lemma 3.2. $\square$

The following lemma directly follows form Corollary 2.4.

Lemma 3.3. *Let Γ be a signed graph with a cut edge $e = uv$, and let $\Gamma - e = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two induced subgraphs of $\Gamma - e$ containing u and v, respectively. Then $\eta(\Gamma) \geq \eta(\Gamma_1) + \eta(\Gamma_2) - 1$.*

Next, we investigate how $\eta(\Gamma)$ changes when we delete a cut vertex from a signed graph Γ. Recall that if w is a cut vertex of Γ, then Γ can be seen as a signed graph obtained by a coalescence of two signed graphs Γ_1 and Γ_2, i.e., $\Gamma = \Gamma_1 \cdot \Gamma_2$.

Lemma 3.4. *If Γ is a signed graph with a cut vertex w such that $\Gamma = \Gamma_1 \cdot \Gamma_2$. Then $\eta(\Gamma) = \eta(\Gamma_1) + \eta(\Gamma_2) - 1$.*

Proof. Without loss of generality, assume that the orders of Γ_1 and Γ_2 are n_1 and n_2, respectively. By arranging the vertices of Γ appropriately, we can write $L^{\pm}(\Gamma)$ as

$$\begin{array}{cc} & \begin{array}{ccc} \phantom{\mathbf{0}_{(n_2-1)\times(n_1-1)}} & w & \phantom{\mathbf{0}_{(n_1-1)\times(n_2-1)}} \end{array} \\ \begin{array}{c} \\ w \\ \\ \end{array} & \begin{bmatrix} B_1 & \alpha_1 & \mathbf{0}_{(n_1-1)\times(n_2-1)} \\ \alpha_1^{\top} & d^{\pm}_{\Gamma}(w) & \alpha_2^{\top} \\ \mathbf{0}_{(n_2-1)\times(n_1-1)} & \alpha_2 & B_2 \end{bmatrix} \end{array},$$

where B_1 and B_2 are the matrices of order $n_1 - 1$ and $n_2 - 1$, respectively, and α_1 and α_2 are the $(n_1 - 1) \times 1$ vector and $(n_2 - 1) \times 1$ vector, respectively. The superscript $\top$ of a matrix represents its transpose. By adding all other rows and columns to the row and column indexed by w, respectively, we obtain a matrix

$$\begin{bmatrix} B_1 & \mathbf{0}_{(n_1-1)\times 1} & \mathbf{0}_{(n_1-1)\times(n_2-1)} \\ \mathbf{0}_{1\times(n_1-1)} & 0 & \mathbf{0}_{1\times(n_2-1)} \\ \mathbf{0}_{(n_2-1)\times(n_1-1)} & \mathbf{0}_{(n_2-1)\times 1} & B_2 \end{bmatrix}.$$

Note that $r(B_1) = r(\Gamma_1)$ and $r(B_2) = r(\Gamma_2)$. Thus, we have $r(\Gamma) = r(B_1) + r(B_2) = r(\Gamma_1) + r(\Gamma_2)$, which implies that $\eta(\Gamma) = n + 1 - r(\Gamma_1) - r(\Gamma_2) - 1 = \eta(\Gamma_1) + \eta(\Gamma_2) - 1$. $\square$

With the help of Lemmas 3.1 and 3.4, we present a corollary below that can simplify the structures of the signed graphs we consider.

Corollary 3.5. *If $\Gamma = \Gamma_1 \cdot T$ is a signed graph and T is a signed tree, then $\eta(\Gamma) = \eta(\Gamma_1)$.*

It is clear from Corollary 3.5 that, when we consider the nullity of a signed graph $\Gamma = \Gamma_1 \cdot T_1 \cdot T_2 \cdot \cdots \cdot T_k$, where $T_1, T_2, \cdots, T_k$ are signed trees, we only need to determine the nullity of Γ_1. So in the remainder of this paper, we can always assume that the signed graph with no pendent signed trees.

Now we are in a position to prove the main result.

Proof of Theorem 1.2. For convenience we abbreviate $\beta(\Gamma)$ as β and choose edges $e_1, \cdots, e_\beta$ from Γ such that $T = \Gamma - \{e_1, \cdots, e_\beta\}$ is a signed tree. Then by Corollary 2.4, we have

$$\begin{gathered} \eta(\Gamma) - 1 \leq \eta(\Gamma - e_1), \\ \eta(\Gamma) - 2 \leq \eta(\Gamma - e_1) - 1 \leq \eta(\Gamma - \{e_1, e_2\}), \\ \vdots \\ \eta(\Gamma) - \beta \leq \eta(\Gamma - \{e_1, \cdots, e_{\beta-1}\}) - 1 \leq \eta(\Gamma - \{e_1, \cdots, e_\beta\}). \end{gathered} \tag{3.1}$$

From the inequalities of (3.1), we obtain $\eta(\Gamma) \leq \eta(T) + \beta = 1 + \beta$. By Lemma 2.1 (2), there do not exist any n-vertices connected signed graphs with nullity n, which means $\eta(\Gamma) \leq n - 1$. So we obtain the inequalities of (i).

When $\eta(\Gamma) = n - 1$ (i.e., $r(\Gamma) = 1$), if there exists an element l_{ij} of $L^{\pm}(\Gamma)$ equal to 0, then all the elements in the same row and column with it are 0s, which is contrary to the fact Γ is connected. Thus, Γ is a signed complete graph and so $d^{\pm}_{\Gamma}(v_i) \in \{-1, 1\}$, where $d^{\pm}_{\Gamma}(v_i)$ is the net-degree of any vertex v_i in Γ. Without loss of generality, assume that there exists a vertex v_1 in Γ with $d^{\pm}_{\Gamma}(v_1) = 1$ (if not, we can consider the signed graph $-\Gamma$ instead of Γ), which means that $d^{+}_{\Gamma}(v_1) = n/2$ and $d^{-}_{\Gamma}(v_1) = n/2 - 1$. Thus n is an even number. By arranging the vertices of Γ appropriately, we denote by $v_2, \cdots, v_{\frac{n}{2}}$ the all negative neighbors of v_1. So far we have determined the elements in the row and column indexed by v_1 in $L^{\pm}(\Gamma)$. Using the condition $r(\Gamma) = 1$, we can write $L^{\pm}(\Gamma)$ as

$$\begin{array}{c} \\ v_1 \\ v_2 \\ \vdots \\ v_{\frac{n}{2}} \\ v_{\frac{n}{2}+1} \\ \vdots \\ v_{n-1} \\ v_n \end{array}
\begin{array}{c}
\begin{array}{cccccccc} v_1 & v_2 & \cdots & v_{\frac{n}{2}} & v_{\frac{n}{2}+1} & \cdots & v_{n-1} & v_n \end{array} \\
\begin{bmatrix}
1 & 1 & \cdots & 1 & -1 & \cdots & -1 & -1 \\
1 & 1 & \cdots & 1 & -1 & \cdots & -1 & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 1 & -1 & \cdots & -1 & -1 \\
-1 & -1 & \cdots & -1 & 1 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-1 & -1 & \cdots & -1 & 1 & \cdots & 1 & 1 \\
-1 & -1 & \cdots & -1 & 1 & \cdots & 1 & 1
\end{bmatrix}
\end{array}.$$

So $\Gamma = K_{\frac{n}{2}} \nabla^{-} K_{\frac{n}{2}}$ or $-(K_{\frac{n}{2}} \nabla^{-} K_{\frac{n}{2}})$. This shows the necessity of (ii) and the sufficiency is obvious.

Assume that Γ is a signed cactus graph which contains no pendent trees in the rest of proof.

For (iii), the assertion follows from Lemmas 3.1 and 3.2 when $\beta = 0, 1$. Therefore, we divide the proof into two cases in which Case 2 will rely on the induction on the cyclomatic number of Γ and Case 1 follows from a direct computation.

Case 1: Γ has no cut edges.

Assume that Γ is obtained by a series of coalescence of the cycles $C_1, \cdots, C_\beta$. By Lemma 3.4 we have $\eta(\Gamma) = \eta(C_1) + \cdots + \eta(C_\beta) - \beta + 1 = \beta - \beta + 1 = 1$, where the second equality is from Lemma 3.2 and $m^{+}(C_i) \neq m^{-}(C_i)$ for each $i = 1, \cdots, \beta$.

In the latter case, we assume that the assertion holds for signed cactus graphs with cyclomatic number at most $\beta - 1$. Recall that $\eta(\Gamma) = 1$ if and only if the coefficient c_1 of the linear term of $P_{L^{\pm}(\Gamma)}(x)$ does not equal to zero.

Case 2: Γ has a cut edge $e = uv$.

Denote the signed graph $\Gamma - e$ as $\Gamma_1 \cup \Gamma_2$ such that $1 \leq \beta(\Gamma_1), \beta(\Gamma_2) \leq \beta - 1$. Since each cycle contained in Γ_1 or Γ_2 also has distinct numbers of positive and negative edges, we have $\eta(\Gamma_1) = 1 = \eta(\Gamma_2)$ by the inductive hypothesis. Thus,

$$\sum_{F^1(\Gamma_1) \in \mathcal{F}^1(\Gamma_1)} \sigma(F^1(\Gamma_1)) \neq 0, \quad \sum_{F^1(\Gamma_2) \in \mathcal{F}^1(\Gamma_2)} \sigma(F^1(\Gamma_2)) \neq 0.$$

Since e is a cut edge, any spanning tree of Γ must contain edge e. Then we have

$$\mathcal{F}^1(\Gamma) = \bigcup_{F^1(\Gamma_1) \in \mathcal{F}^1(\Gamma_1)} \bigcup_{F^1(\Gamma_2) \in \mathcal{F}^1(\Gamma_2)} \{F^1(\Gamma_1) \cup F^1(\Gamma_2) + e\},$$

where $F^1(\Gamma_1) \cup F^1(\Gamma_2) + e$ is obtained by adding an edge e to $F^1(\Gamma_1) \cup F^1(\Gamma_2)$, and so

$$\sum_{F^1(\Gamma)\in\mathcal{F}^1(\Gamma)} \sigma(F^1(\Gamma)) = (\sum_{F^1(\Gamma_1)\in\mathcal{F}^1(\Gamma_1)} \sigma(F^1(\Gamma_1))) \cdot \sigma(e) \cdot (\sum_{F^1(\Gamma_2)\in\mathcal{F}^1(\Gamma_2)} \sigma(F^1(\Gamma_2))).$$

Thus, $c_1 \neq 0$ if and only if both $(\sum_{F^1(\Gamma_1)\in\mathcal{F}^1(\Gamma_1)} \sigma(F^1(\Gamma_1)))$ and $(\sum_{F^1(\Gamma_2)\in\mathcal{F}^1(\Gamma_2)} \sigma(F^1(\Gamma_2)))$ do not equal to zero, that is, $c_1 \neq 0$ if and only if both Γ_1 and Γ_2 are signed cactus graphs in which each cycle has distinct numbers of positive and negative edges. The cycles of Γ consist of that of Γ_1 and Γ_2, so by inductive hypothesis we complete the proof in this case.

If $\eta(\Gamma) = \beta + 1$, suppose to the contrary that there exists a signed cycle C in Γ such that $m^+(C) \neq m^-(C)$. We take $\beta - 1$ edges $e_1, \cdots, e_{\beta-1}$ in Γ such that $U = \Gamma - \{e_1, \cdots, e_{\beta-1}\}$ is a signed unicyclic graph with the unique cycle C. By Corollary 2.4 and Lemma 3.2 we obtain $2 = \eta(\Gamma) - \beta + 1 \leq \eta(\Gamma - \{e_1, \cdots, e_{\beta-1}\}) = 1$, a contradiction. This shows the necessity of (iv).

In what follows, we will prove $\eta(\Gamma) = \beta + 1$, where Γ is a signed cactus graph in which any cycle C has $m^+(C) = m^-(C)$. We also divide our proof into two parts according to two cases.

Case 1: Γ has no cut edges.
Assume that $C_1, \cdots, C_\beta$ are all cycles in Γ. In fact, Γ can be obtained by a series of coalescence of these cycles. So from Lemma 3.4 we have $\eta(\Gamma) = \eta(C_1) + \cdots + \eta(C_\beta) - \beta + 1 = 2\beta - \beta + 1 = \beta + 1$, where the second equality is from Lemma 3.2 and $m^+(C_i) = m^-(C_i)$ for each $i = 1, \cdots, \beta$.

In the remaining case, we proceed by induction on the cyclomatic number of Γ. Assume that the assertion holds for all signed cactus graphs with cyclomatic number at most $\beta - 1$ and let Γ be a signed cactus graph with cyclomatic number β.

Case 2: Γ has a cut edge $e = uv$.
Denote the signed graph $\Gamma - e$ as $\Gamma_1 \cup \Gamma_2$ such that $1 \leq \beta(\Gamma_1), \beta(\Gamma_2) \leq \beta - 1$, where $\beta(\Gamma_1)$ and $\beta(\Gamma_2)$ are the cyclomatic numbers of Γ_1 and Γ_2, respectively. Note that each cycle contained in Γ_1 or Γ_2 also has the equal number of positive and negative edges and $\beta(\Gamma) = \beta(\Gamma_1) + \beta(\Gamma_2)$. By Lemma 3.3 and inductive hypothesis we have

$$\eta(\Gamma) \geq \eta(\Gamma_1) + \eta(\Gamma_2) - 1 = \beta(\Gamma_1) + 1 + \beta(\Gamma_2) + 1 - 1 = \beta(\Gamma) + 1.$$

Combining this with $\eta(\Gamma) \leq \beta + 1$ we obtain the conclusion in this case. □

With the help of Theorem 1.2, for a signed cactus graph Γ, we study how $\eta(\Gamma)$ changes when we delete an edge of them.

Corollary 3.6. *Let Γ be a signed cactus graph with cyclomatic number β and e be an edge of any cycle in Γ.*

(1) *If $m^+(C) = m^-(C)$ for any cycle C of Γ, then $\eta(\Gamma - e) = \eta(\Gamma) - 1$.*

(2) *If $m^+(C) \neq m^-(C)$ for any cycle C of Γ, then $\eta(\Gamma - e) = \eta(\Gamma) = 1$.*

Proof. As we have shown in the proof of Theorem 1.2, all the inequalities of (3.1) become equalities when $\eta(\Gamma) = \beta + 1$, which leads to (1).

For (2), Since $\Gamma - e$ is also a signed cactus graph in which any cycle has distinct numbers of positive and negative edges, then by Theorem 1.2 we have $\eta(\Gamma - e) = 1 = \eta(\Gamma)$. □

4 Concluding remarks

At the end of this paper, it is important to note that we only consider signed cactus graphs that achieve the nullity 1 or $\beta(\Gamma)+1$. For signed graphs in which two cycles share common edges, the discussion appears to be more complex. For instance, the nullity of the signed graph shown in Figure 1 is 1, but the cycles 12361 and 34563 have 2 positive edges and 2 negative edges. Thus, the result of Theorem 1.2 (*iii*) does not hold for signed graph in general. In conclusion, the following proposition addresses the nullity of a signed bicyclic graph where two cycles share a common edge.

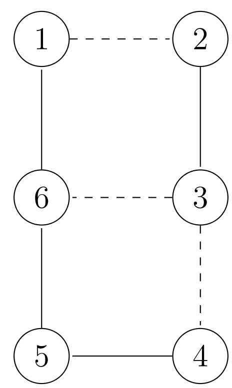

Figure 1: A signed graph of nullity 1. The positive edges (resp., negative edges) are presented by solid lines (resp., dashed lines).

Bear in mind that we just need to consider the signed graphs with no pendant trees.

Proposition 4.1. Let Γ be a bicyclic signed graph of order n where two cycles C_1 and C_2 share a common edge e, that is, $\Gamma = P_1 \cup e \cup P_2$ where $P_1 \cup e = C_1$ and $P_2 \cup e = C_2$. Then $\eta(\Gamma) \leq 2$. Moreover, $\eta(\Gamma) = 2$ if and only if $m^+(C_1) - m^-(C_1) = m^+(C_2) - m^-(C_2) = \pm 1$.

Proof. Without loss of generality, assume that e is a positive edge. It is from Theorem 1.2 that $\eta(\Gamma) \leq \beta(\Gamma) + 1 = 3$. Denote by C_3 the cycle formed by P_1 and P_2, that is, $C_3 = P_1 \cup P_2$. Assume for a contradiction that $\eta(\Gamma) = 3$. Then it follows from Theorem 1.2 that $m^+(C_i) = m^-(C_i)$ for $i = 1, 2, 3$. Then we have

$$\begin{cases} m^-(P_1) - m^+(P_1) = 1 \\ m^-(P_2) - m^+(P_2) = 1 \\ m^+(P_1) + m^+(P_2) = m^-(P_1) + m^-(P_2). \end{cases}$$

From these equations we derive a contradiction, obtaining that $\eta(\Gamma) \leq 2$. Combining this with Theorem 1.1, $\eta(\Gamma) = 2$ if and only if $c_1 = (-1)^{n-1} n (\sum_{F^1(\Gamma) \in \mathcal{F}^1(\Gamma)} \sigma(F^1(\Gamma))) = 0$. The spanning trees in $\mathcal{F}^1(\Gamma)$ can be obtained by deleting one edge of P_1 and one edge of P_2 or by deleting the edge e and one edge of $P_1 \cup P_2$. Thus, we have

$$\begin{aligned}\sum_{F^1(\Gamma)\in\mathcal{F}^1(\Gamma)} \sigma(F^1(\Gamma)) &= [m^+(C_1) - 1 - m^-(C_1)]\sigma(C_1)[m^+(C_2) - 1 - m^-(C_2)]\sigma(C_2) \\ &\quad + m^+(C_3)\sigma(C_3) - m^-(C_3)\sigma(C_3) \\ &= (m^\pm(C_1) - 1)(m^\pm(C_2) - 1)\sigma(C_3) + m^\pm(C_3)\sigma(C_3) \\ &= (m^\pm(C_1)m^\pm(C_2) - 1)\sigma(C_3),\end{aligned}$$

where $m^\pm(C_i) = m^+(C_i) - m^-(C_i)$ for $i = 1, 2$. Then $\sum_{F^1(\Gamma)\in\mathcal{F}^1(\Gamma)} \sigma(F^1(\Gamma)) = 0$ if and only if $m^\pm(C_1)m^\pm(C_2) = 1$. The case of $\sigma(e) = -1$ can be proven similarly. This completes the proof of Proposition 4.1. □

Proposition 4.1 provides valuable insights into the nullity of bicyclic signed graphs. However, it is worth noting that the technique used in proving this proposition is specific to the case of bicyclic signed graphs where two cycles share a common edge. For signed graphs with a cyclomatic number greater than 1, this technique may not be straightforward to apply. Therefore, further study on characterizing all the signed graphs with nullity 1 or $\beta(\Gamma) + 1$ may require the development and application of new methods.

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